NEROGRAV

New Refined Observations of Climate Change from Spaceborne Gravity Missions

International Spring School
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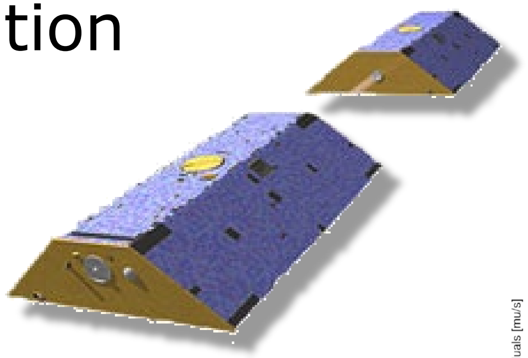
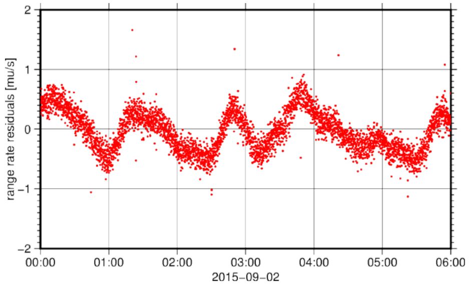
Stochastic Modeling of GRACE/GRACE-FO Data

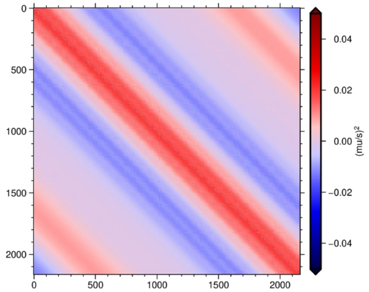
Michael Murböck (TU Berlin)

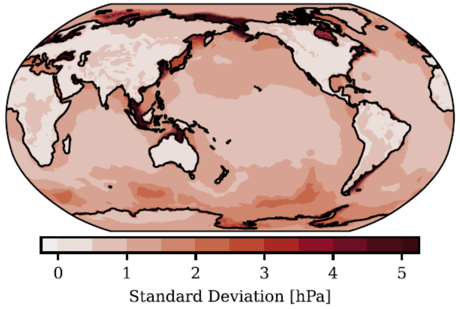
Overview

- From measurements to gravity field solution
  - Least squares adjustment
- Residual analysis
- Weight matrix of instrument noise
- Background model noise
  - Example: non-tidal AOD model
- Summary





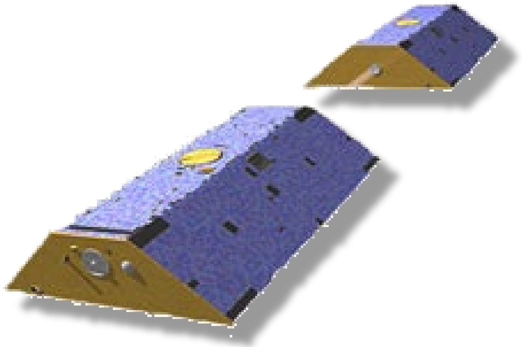




From measurements to gravity field solution

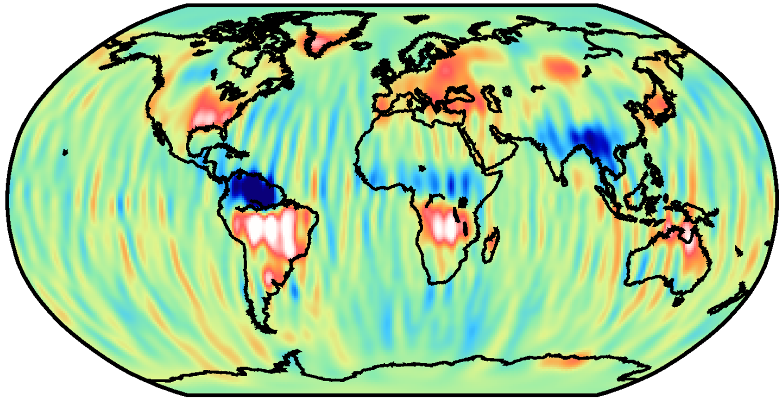
Measurements

- Range-rate data between the satellites
- Accelerometer data
- Star camera data
- …



Monthly gravity field of the Earth

- Spherical harmonics coefficients
- or Mascons
- or …

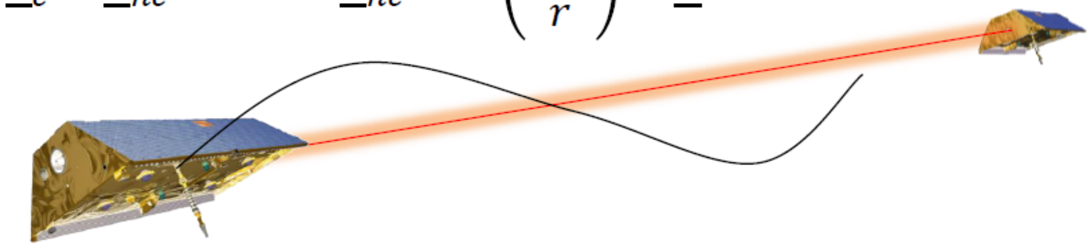


From measurements to gravity field solution

6. GRACE observation equation

Equation of motion

$$\ddot{\underline{r}} = \underline{a}_c + \underline{a}_{nc} = \nabla V + \underline{a}_{nc} = \nabla\left(\frac{GM}{r}\right) + \underline{d}$$



Observations:
- (biased) range ρ
- range rate $\dot\rho$
- range acceleration $\ddot\rho$

Numerical orbit integration → position+velocity → range/range rate

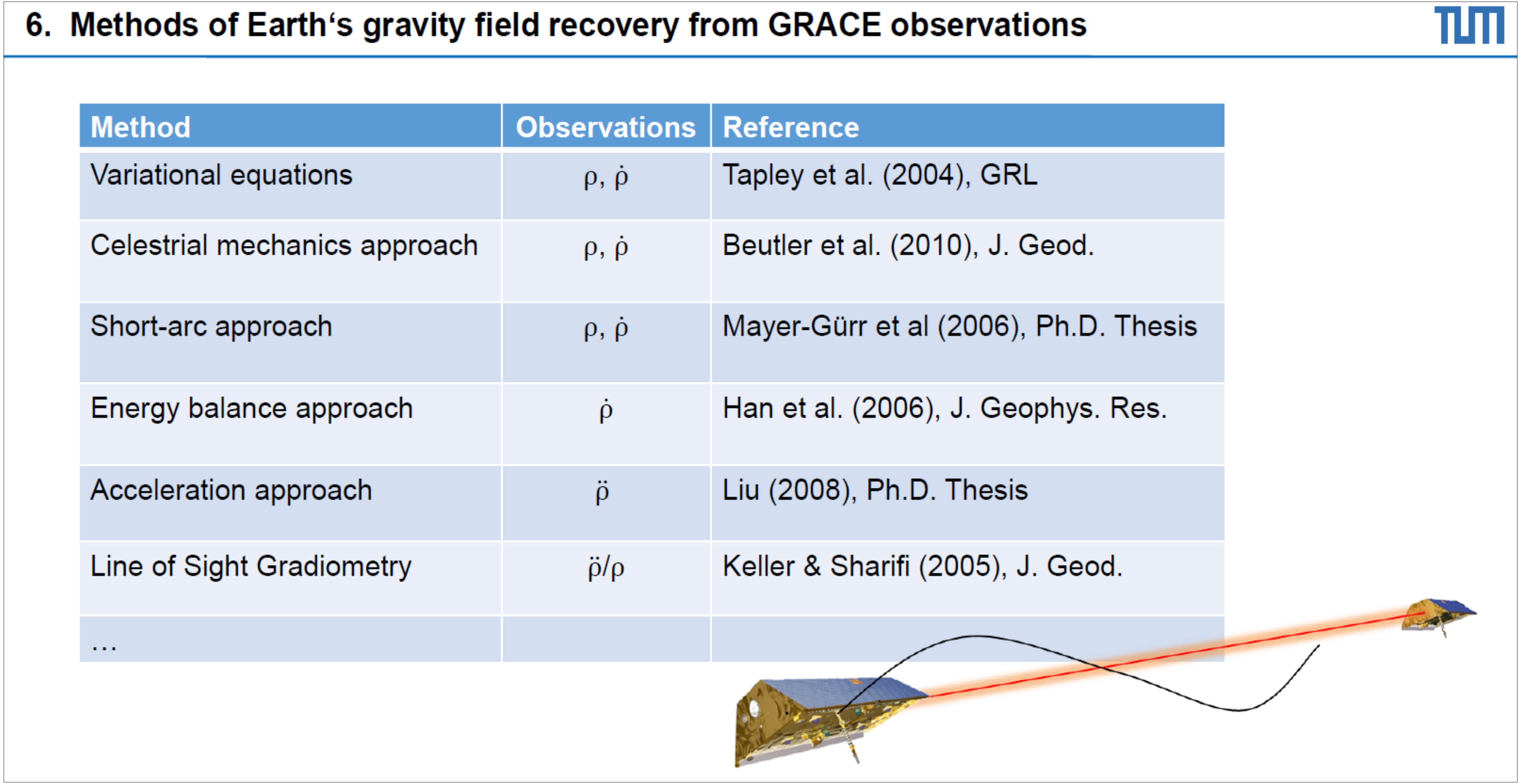
Observation equation

$$y(t) = f\Big(t, \overbrace{\underline{r}, \underline{\dot r}}^{\text{Orbit}}, \underset{\substack{\uparrow \\ \text{Gravity field} \\ \text{parameters}}}{\underline{x}}, \overset{\substack{\text{Force models} \\ \text{(ACC meas.)} \\ \downarrow}}{\underline{d}}, \underbrace{cal, emp}_{\substack{\text{Calibration/empir.} \\ \text{parameters}}}, \dots\Big) = f_0\Big(t, \underbrace{\underline{r}_0, \underline{\dot r}_0, \underline{x}_0}_{\text{A-priori models}}, \dots\Big) + \underset{\substack{\uparrow \\ \text{Obs. Residuals} \\ \rightarrow \text{input to adjustment}}}{\delta f}$$

$$\underline{x} = \{\bar{C}_{nm}, \bar{S}_{nm}\}$$

From Level-1B Instrument Data to Level-2 Spherical Harmonics (Gruber, TUM)

From measurements to gravity field solution

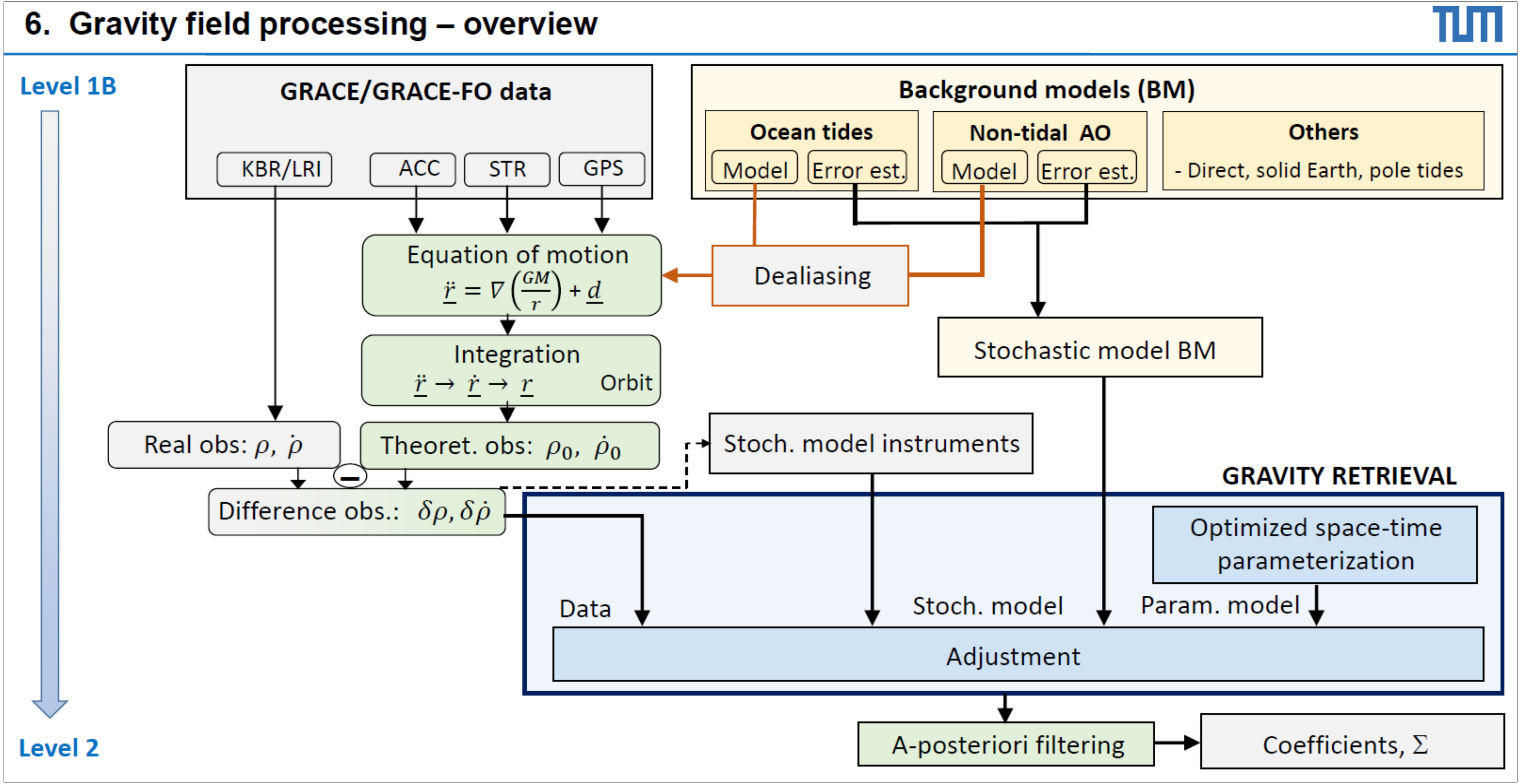


6. Methods of Earth‘s gravity field recovery from GRACE observations

Method	Observations	Reference
Variational equations	$\rho, \dot{\rho}$	Tapley et al. (2004), GRL
Celestrial mechanics approach	$\rho, \dot{\rho}$	Beutler et al. (2010), J. Geod.
Short-arc approach	$\rho, \dot{\rho}$	Mayer-Gürr et al (2006), Ph.D. Thesis
Energy balance approach	$\dot{\rho}$	Han et al. (2006), J. Geophys. Res.
Acceleration approach	$\ddot{\rho}$	Liu (2008), Ph.D. Thesis
Line of Sight Gradiometry	$\ddot{\rho}/\rho$	Keller & Sharifi (2005), J. Geod.
…		

From Level-1B Instrument Data to Level-2 Spherical Harmonics (Gruber, TUM)

From measurements to gravity field solution



6. Gravity field processing – overview

Level 1B

Level 2

GRACE/GRACE-FO data

KBR/LRI

ACC

STR

GPS

Background models (BM)

Ocean tides

Model

Error est.

Non-tidal AO

Model

Error est.

Others

- Direct, solid Earth, pole tides

Equation of motion

$\ddot{\underline{r}} = \nabla\left(\frac{GM}{r}\right) + \underline{d}$

Dealiasing

Integration

$\ddot{\underline{r}} \rightarrow \dot{\underline{r}} \rightarrow \underline{r}$ Orbit

Stochastic model BM

Real obs: $\rho,\ \dot{\rho}$

Theoret. obs: $\rho_0,\ \dot{\rho}_0$

$-$

Stoch. model instruments

Difference obs.: $\delta\rho, \delta\dot{\rho}$

GRAVITY RETRIEVAL

Optimized space-time parameterization

Data

Stoch. model

Param. model

Adjustment

A-posteriori filtering

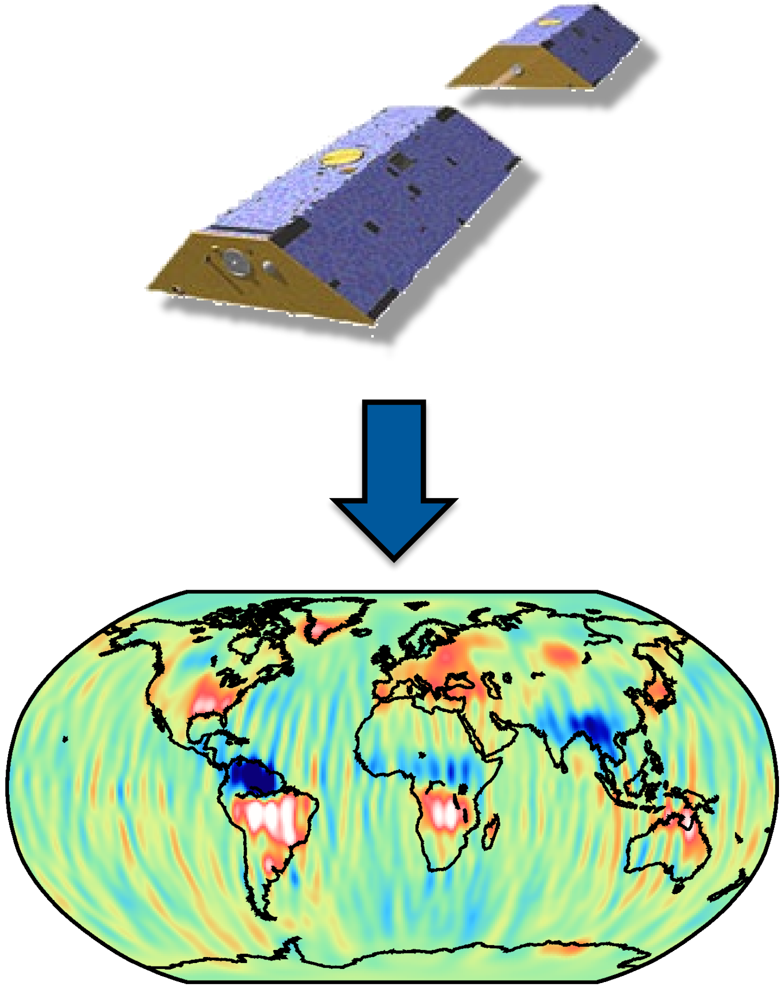
Coefficients, Σ

From Level-1B Instrument Data to Level-2 Spherical Harmonics (Gruber, TUM)

Least squares adjustment

- Prediction of the observations (range rate data) by a model

$$y(t) = f(\quad ???\quad)$$

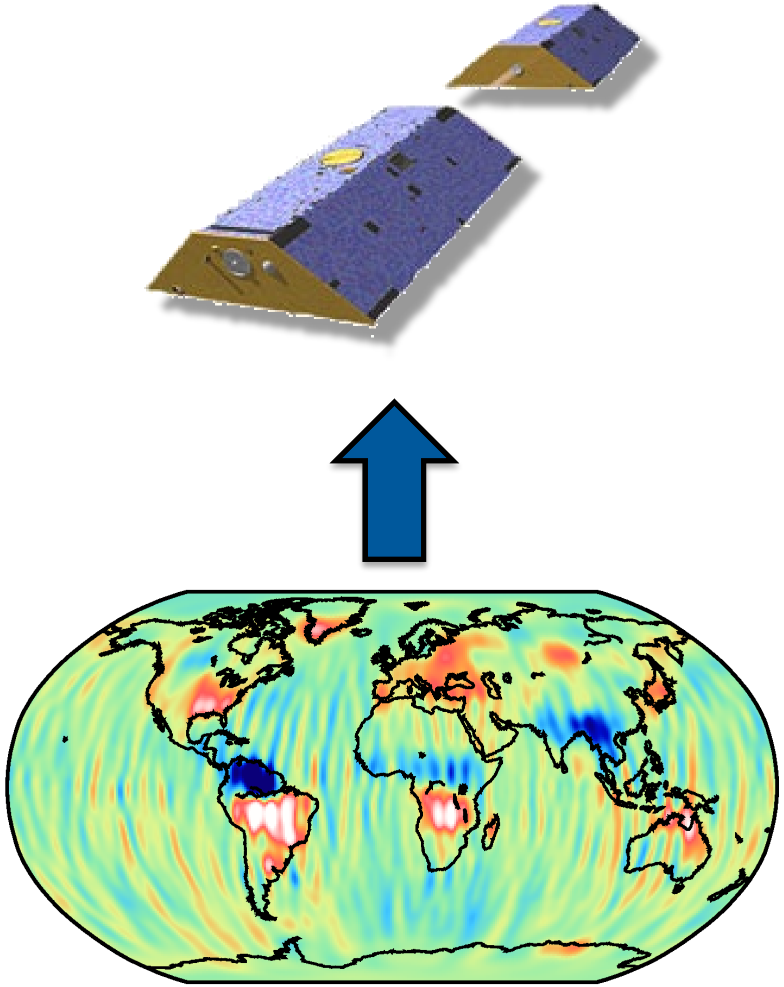


Least squares adjustment

- Prediction of the observations (range rate data) by a model

$$y(t) = f(\quad ???\quad)$$

Observations:
range rate



Least squares adjustment

- Prediction of the observations (range rate data) by a model

$$y(t) = f(t, \mathbf{r}_0, \dot{\mathbf{r}}_0, \mathbf{x}, cal, \dots)$$

Observations:
range rate

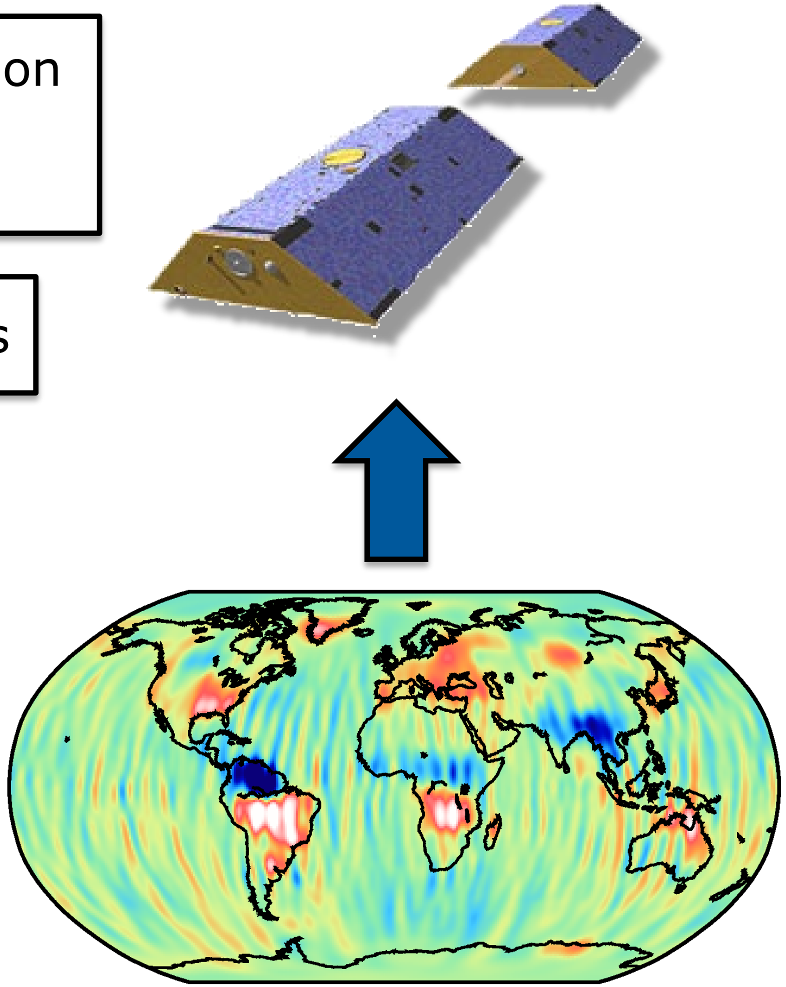
Time

Satellite state

Gravity field parameters

Instrumental calibration
- Accelerometer drift
- ...

Physical model:
- Solution of a differential equation (Orbit integration)
- non linear!



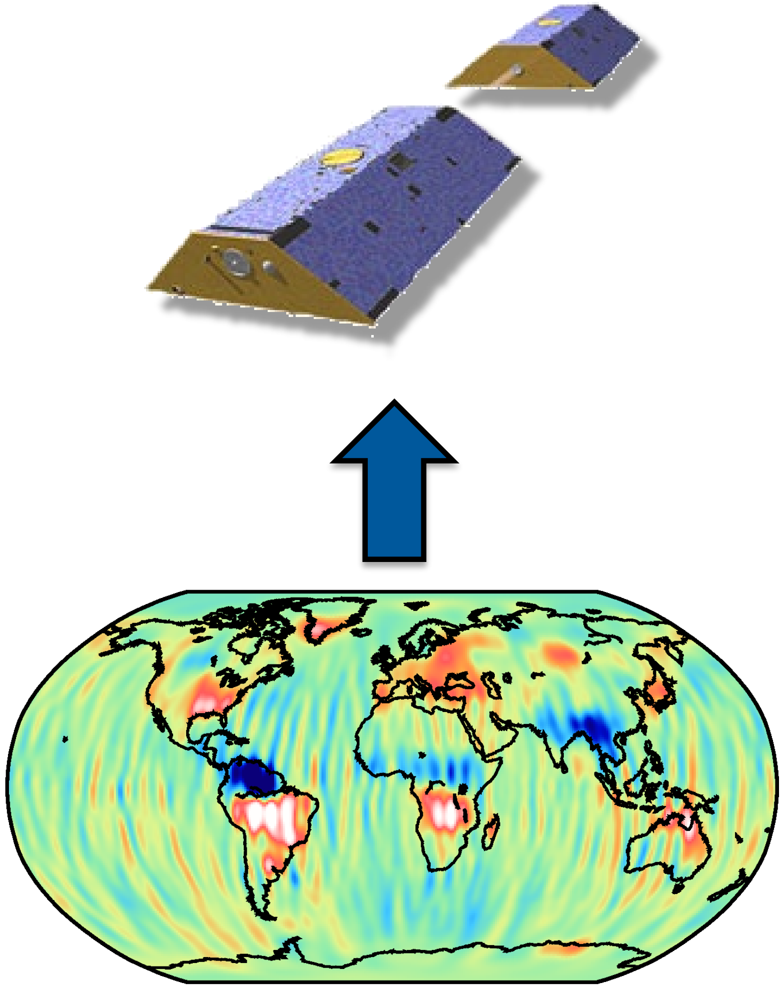
Least squares adjustment

- Prediction of the observations (range rate data) by a model

$$y(t) = f(t, \mathbf{r}_0, \dot{\mathbf{r}}_0, \mathbf{x}, cal, \dots)$$

- Linearization by a truncated Taylor series

$$y = f(\mathbf{x}_0) + \left.\frac{\partial f}{\partial \mathbf{x}}\right|_0 (\mathbf{x} - \mathbf{x}_0) + \cdots$$



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$$y(t) = f(t, \mathbf{r}_0, \dot{\mathbf{r}}_0, \mathbf{x}, cal, \dots)$$

- Linearization by a truncated Taylor series

$$y - f(\mathbf{x}_0) = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_0 (\mathbf{x} - \mathbf{x}_0) + \cdots$$

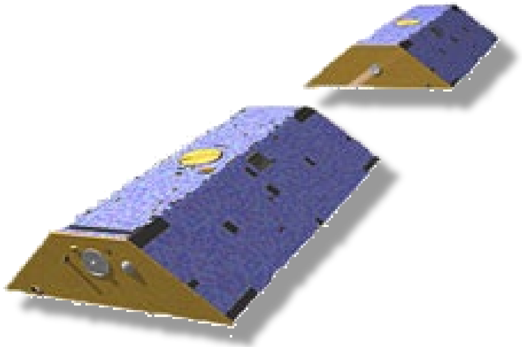
- Linear, overdetermined system of equations (in matrix form)

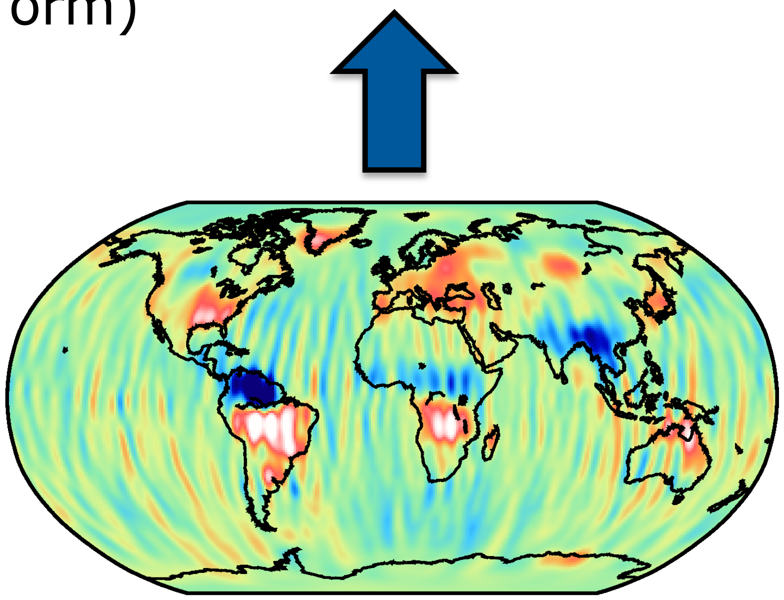
$$\Delta \mathbf{l} = \mathbf{A}\, \Delta \mathbf{x}$$

500,000 observations per month

Fully occupied matrix (~ 100 GB)

15,000 Parameter + Calibration + ...





Least squares adjustment

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- Linear, overdetermined system of equations (in matrix form)

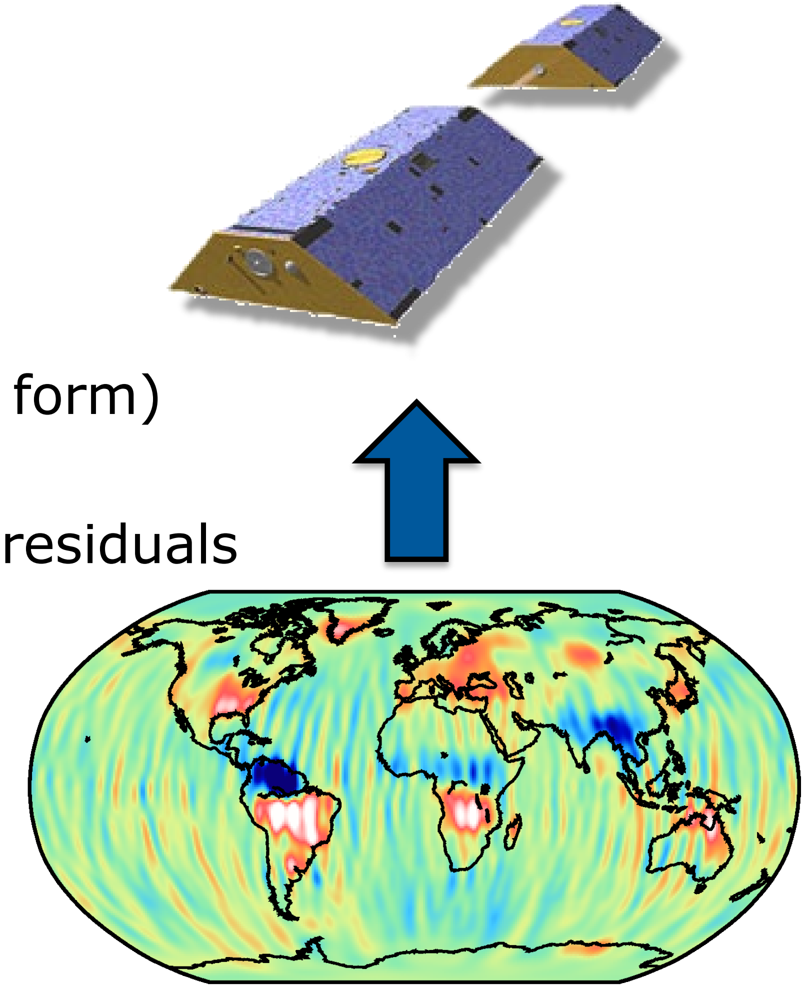
$$\Delta\mathbf{l} = \mathbf{A}\,\Delta\mathbf{x}$$

- Solution by minimizing the weighted quadratic sum of residuals

$$\|\Delta\mathbf{l} - \mathbf{A} \cdot \Delta\mathbf{x}\|_{\mathbf{P}}^2 \to \min$$

- Solution

$$\Delta\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\Delta\mathbf{l}$$



Least squares adjustment

- Prediction of the observations (range rate data) by

$$y(t) = f(t, \mathbf{r}_0, \dot{\mathbf{r}}_0, \mathbf{x}, cal, \dots)$$

- Linearization by a truncated Taylor series

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$$||\Delta\mathbf{l} - \mathbf{A}\cdot\Delta\mathbf{x}||^2_{\mathbf{P}} \rightarrow \min$$

- Solution

$$\Delta\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\Delta\mathbf{l}$$

Reduced observation vector

$$\Delta\mathbf{l} = \big(y - f(\mathbf{x}_0)\big)$$

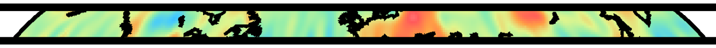
is a mixture of all instruments and models

- Differentiated Ranging Instrument noise
- Integrated Accelerometer noise
- Aliasing errors from background models
  - Atmosphere, ocean mass variations
  - Ocean tides
- …

The the post-fit residuals (How good is the fit?)

$$\hat{\mathbf{e}} = \Delta\mathbf{l} - \mathbf{A}\cdot\Delta\hat{\mathbf{x}}$$

tell us something about the noise



The weight matrix

$$\mathbf{P} = \mathbf{\Sigma}^{-1}$$

should consider the noise of the observation vector

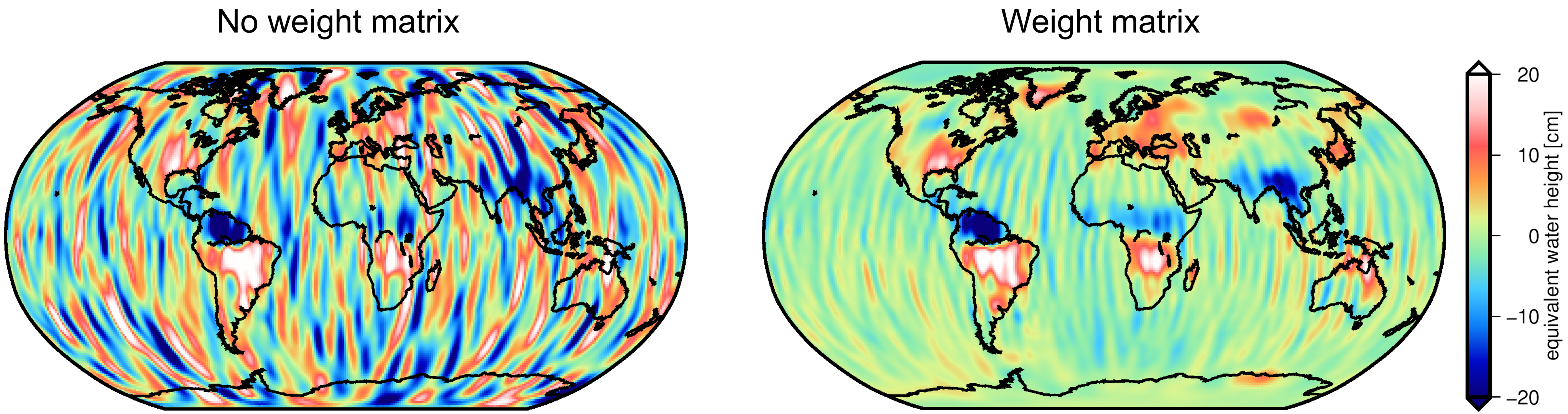
$\Rightarrow$ Inverse of the noise covariance matrix

Least squares adjustment

- Almost every method for gravity field recovery is based on least squares adjustment

$$||\Delta\mathbf{l} - \mathbf{A}\cdot\Delta\mathbf{x}||_{\mathbf{P}}^2 \rightarrow \min$$

- This talk is only about the weight matrix **P**


No weight matrix

Weight matrix

Least squares adjustment

- Almost every method for gravity field recovery is based on least squares adjustment

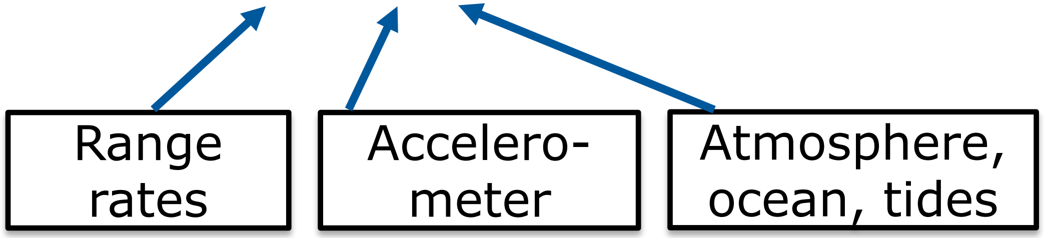
$$||\Delta\mathbf{l} - \mathbf{A} \cdot \Delta\mathbf{x}||_{\mathbf{P}}^2 \rightarrow \min$$

- This talk is only about the weight matrix **P**

- The weight matrix should consider the noise of the reduced observation vector $\mathbf{P} = \mathbf{\Sigma}^{-1}$

- The reduced observation vector contains different noise sources

$$\Delta\mathbf{l} = \left(y - f(\mathbf{x}_0)\right)$$



Range rates

Accelero-meter

Atmosphere, ocean, tides

Least squares adjustment

- Almost every method for gravity field recovery is based on least squares adjustment

$$||\Delta\mathbf{l} - \mathbf{A} \cdot \Delta\mathbf{x}||_{\mathbf{P}}^2 \rightarrow \min$$

- This talk is only about the weight matrix **P**

- The weight matrix should consider the noise of the reduced observation vector $\mathbf{P} = \mathbf{\Sigma}^{-1}$

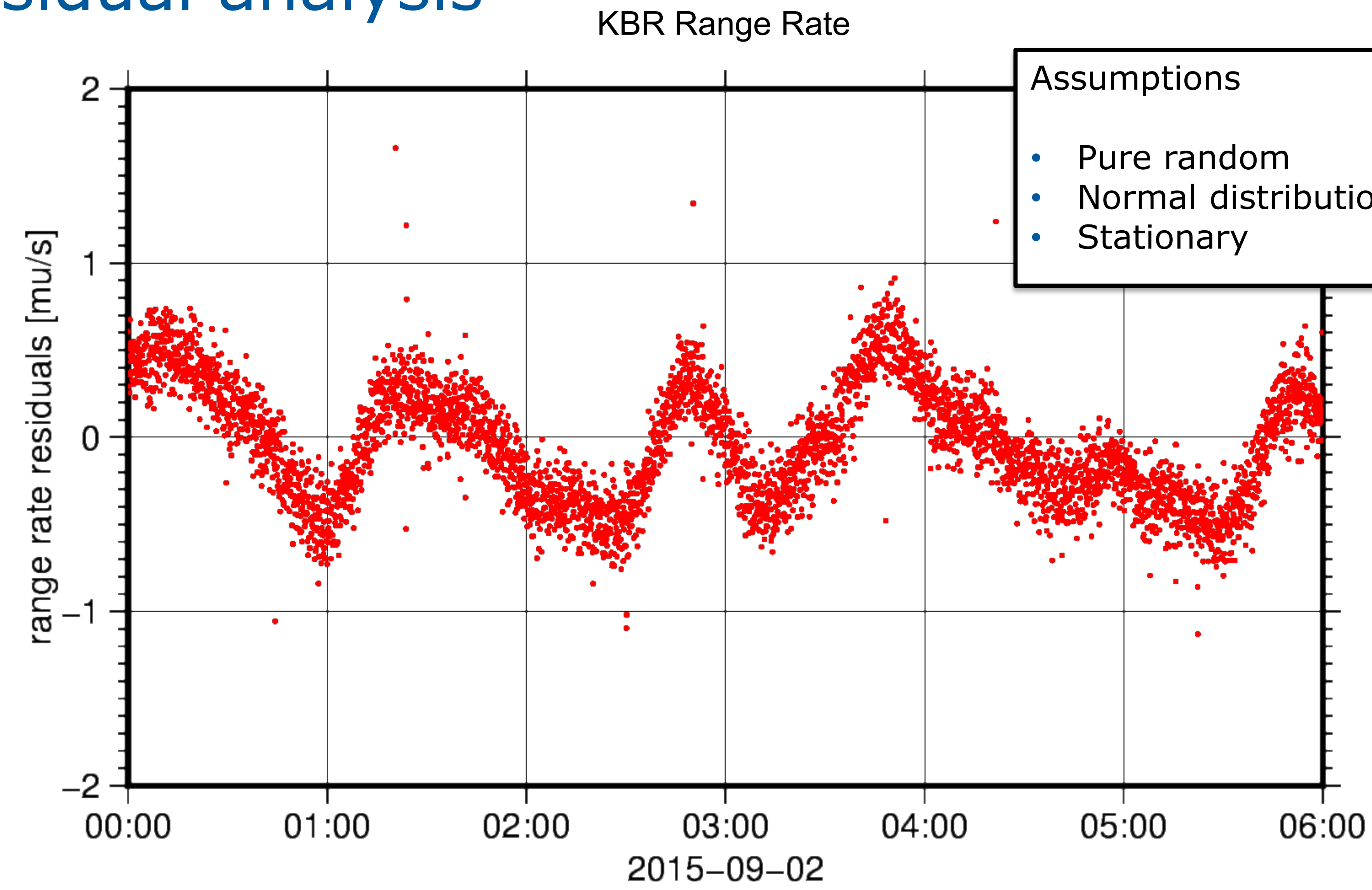
- The reduced observation vector contains different noise sources

$$\Delta\mathbf{l} = \left(y - f(\mathbf{x}_0)\right)$$

- We can analyize the post-fit residuals to understand the noise

$$\hat{\mathbf{e}} = \Delta\mathbf{l} - \mathbf{A} \cdot \Delta\hat{\mathbf{x}}$$

Residual analysis


KBR Range Rate

Assumptions

- Pure random
- Normal distribution
- Stationary

Residual analysis

- Assumption of a stationary noise

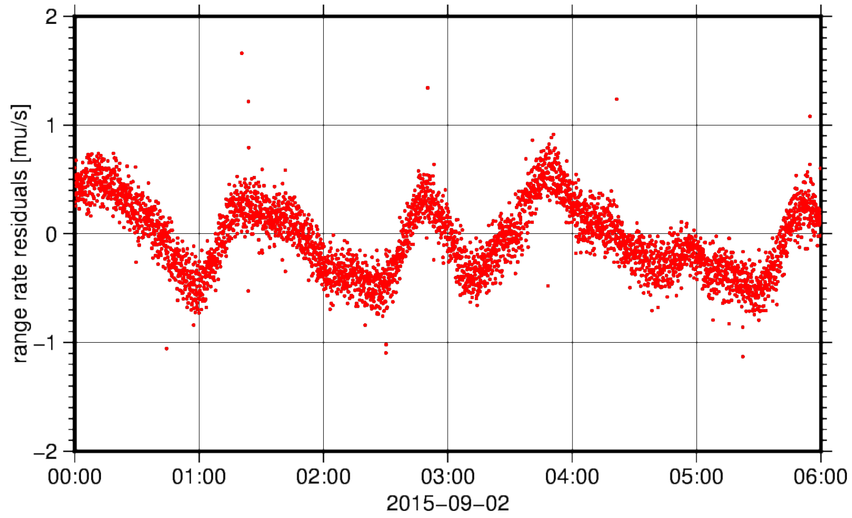
$$RMS^2 = \frac{1}{N}\sum_{i=0}^{N} \hat{e}^2(t_i)$$

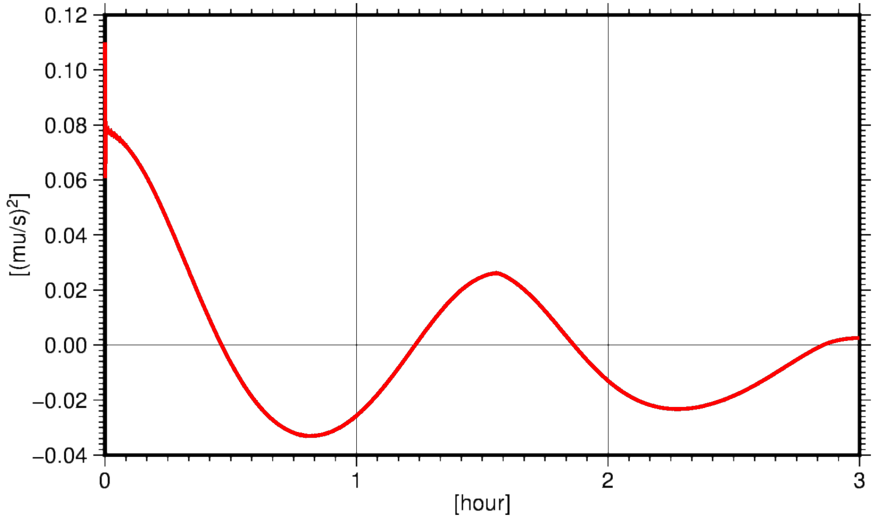
- Neighboring residuals are correlated: Estimation of the covariance

$$\mathrm{cov}(\Delta t_k) = \frac{1}{N}\sum_{i=0}^{N} \hat{e}(t_i)\hat{e}(t_i + \Delta t_k)$$

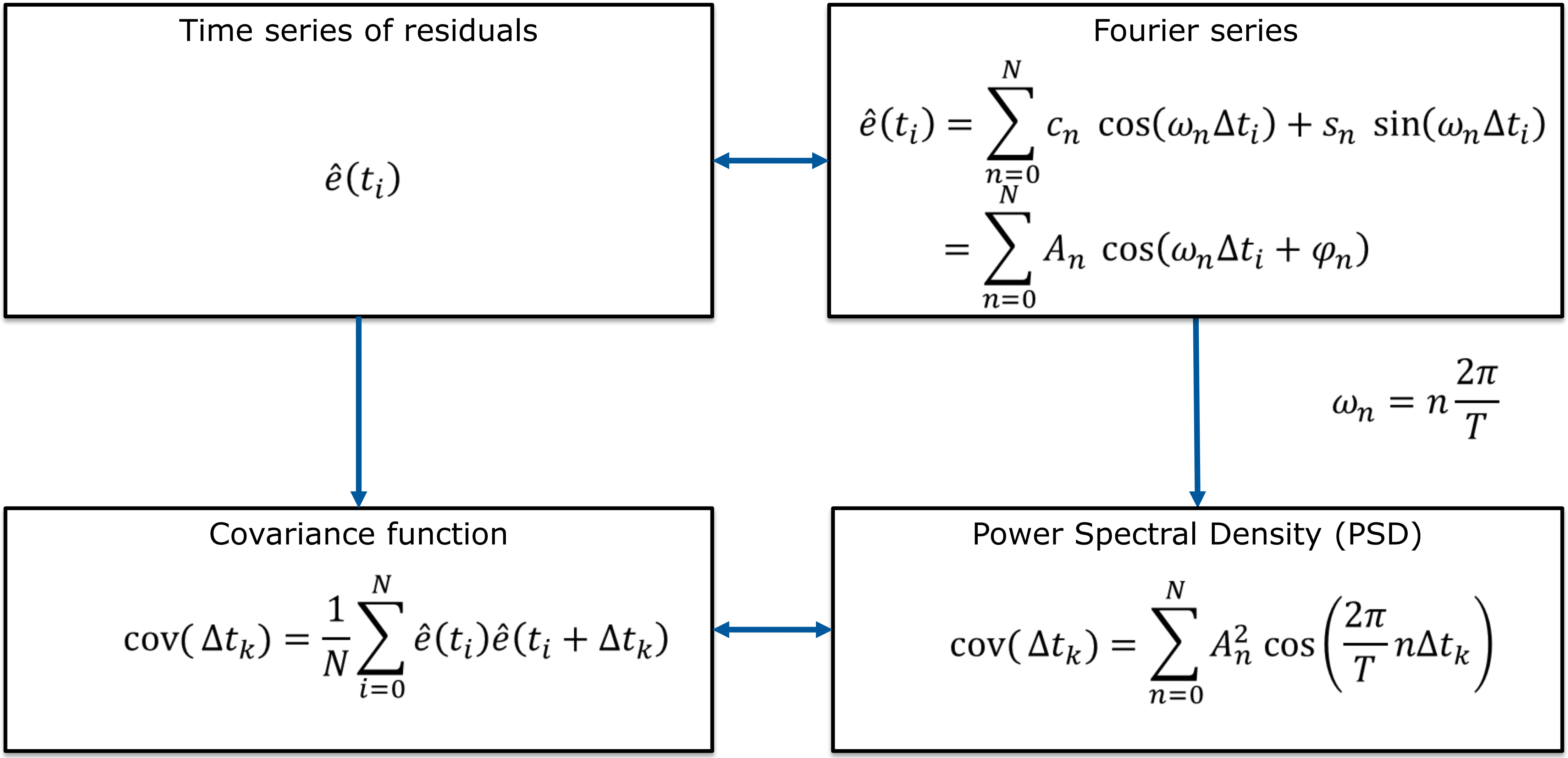
- The covariance function can be expressed by the amplitudes of a power spectrum (PSD)

$$\mathrm{cov}(\Delta t_k) = \sum_{n=0}^{N} a_n^2 \cos\left(\frac{2\pi}{T} n\Delta t_k\right)$$





Residual analysis



Time series of residuals

$$\hat{e}(t_i)$$

Fourier series

$$\hat{e}(t_i) = \sum_{n=0}^{N} c_n \, \cos(\omega_n \Delta t_i) + s_n \, \sin(\omega_n \Delta t_i)$$

$$= \sum_{n=0}^{N} A_n \, \cos(\omega_n \Delta t_i + \varphi_n)$$

$$\omega_n = n \frac{2\pi}{T}$$

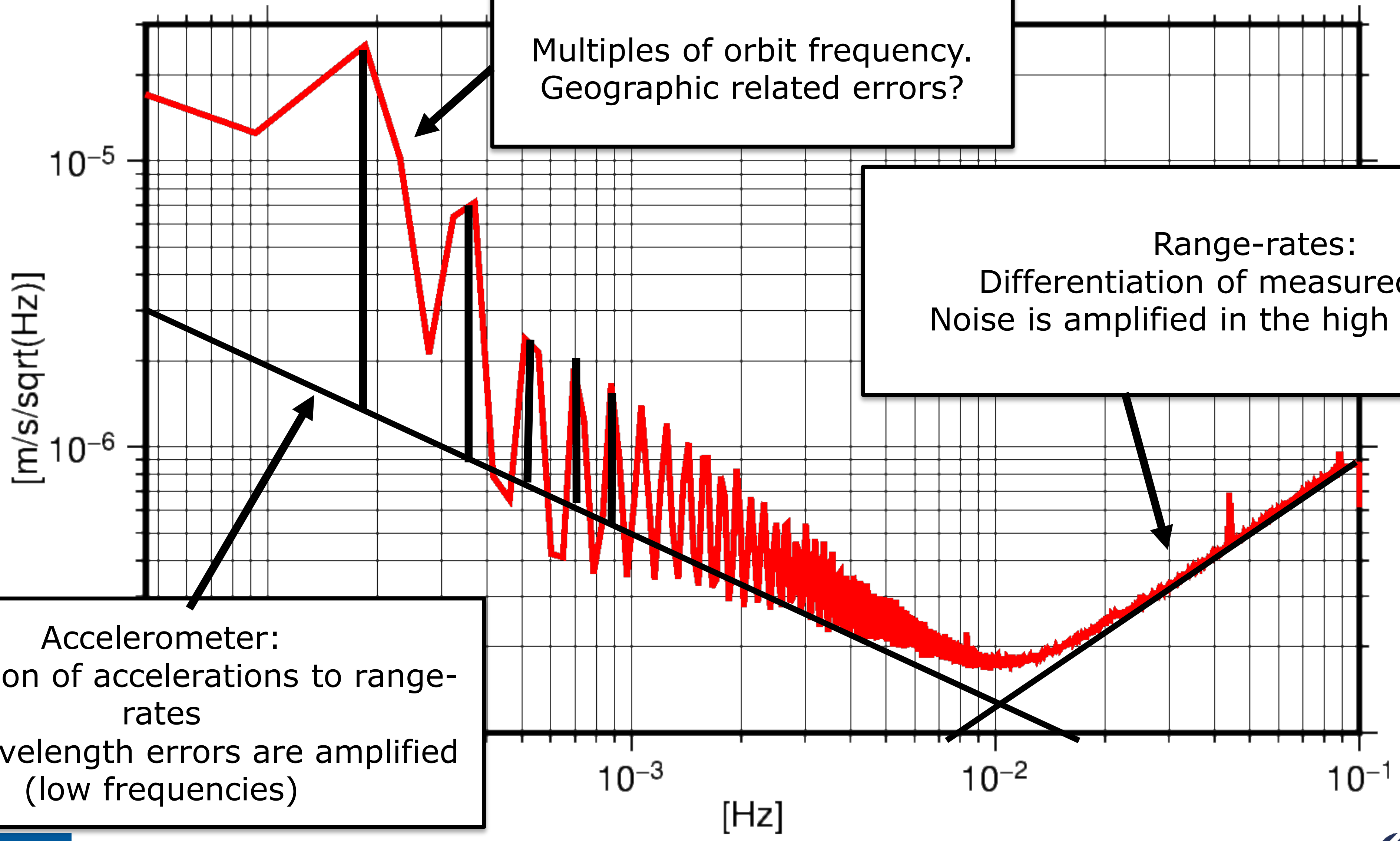
Covariance function

$$\operatorname{cov}(\Delta t_k) = \frac{1}{N} \sum_{i=0}^{N} \hat{e}(t_i) \hat{e}(t_i + \Delta t_k)$$

Power Spectral Density (PSD)

$$\operatorname{cov}(\Delta t_k) = \sum_{n=0}^{N} A_n^2 \cos\left(\frac{2\pi}{T} n \Delta t_k\right)$$

Residual analysis



Multiples of orbit frequency.
Geographic related errors?

Range-rates:
Differentiation of measured ranges
Noise is amplified in the high frequencies

Accelerometer:
Integration of accelerations to range-rates
Long-wavelength errors are amplified (low frequencies)

Weight matrix of instrument noise

- Estimation of the covariance function

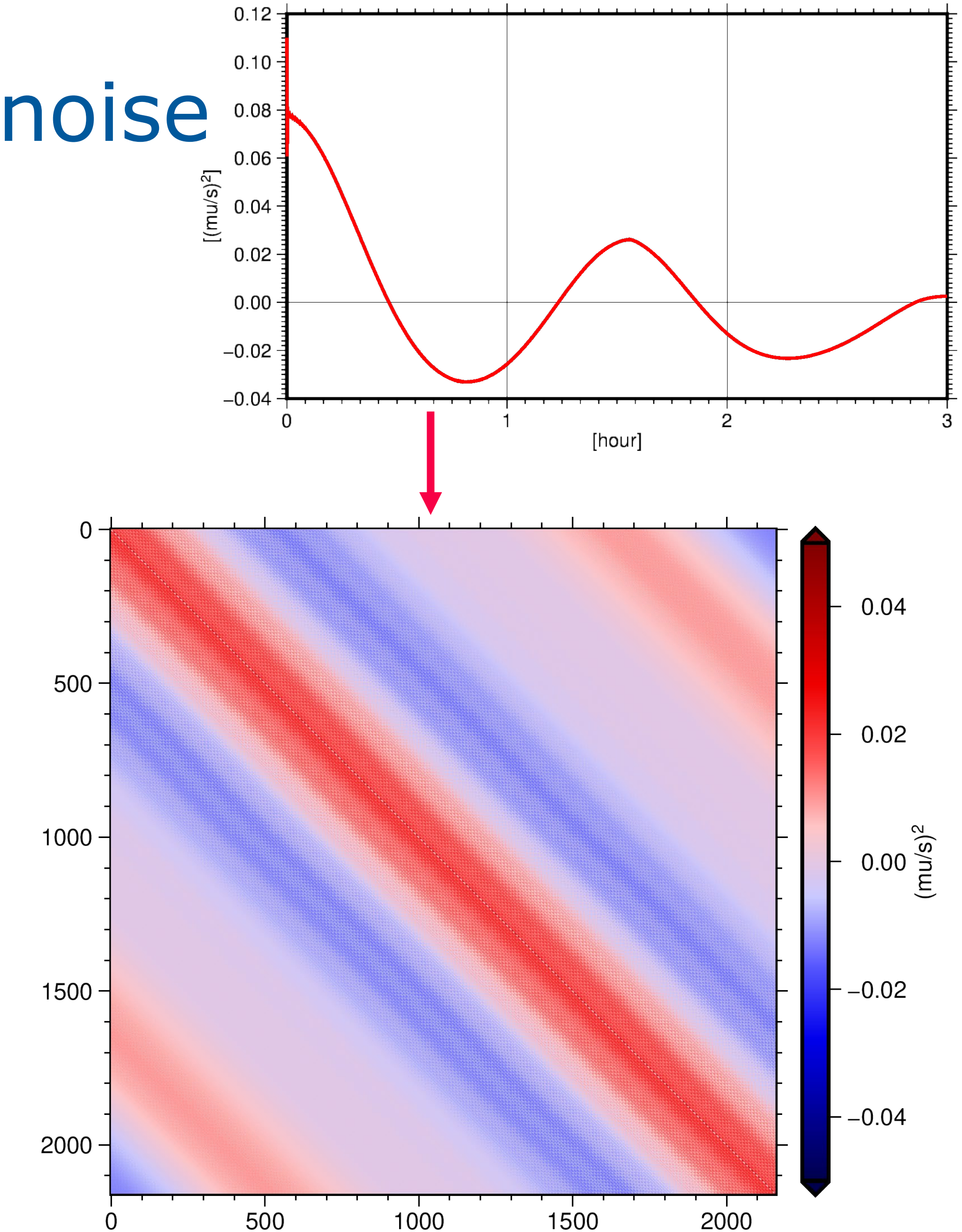
$$\mathrm{cov}(\Delta t_k) = \frac{1}{N}\sum_{i=0}^{N} \hat{e}(t_i)\hat{e}(t_i + \Delta t_k)$$

- Assumption of a stationary noise process:
  - Covariance matrix is a Toeplitz matrix
  - Can be described by the covariance function

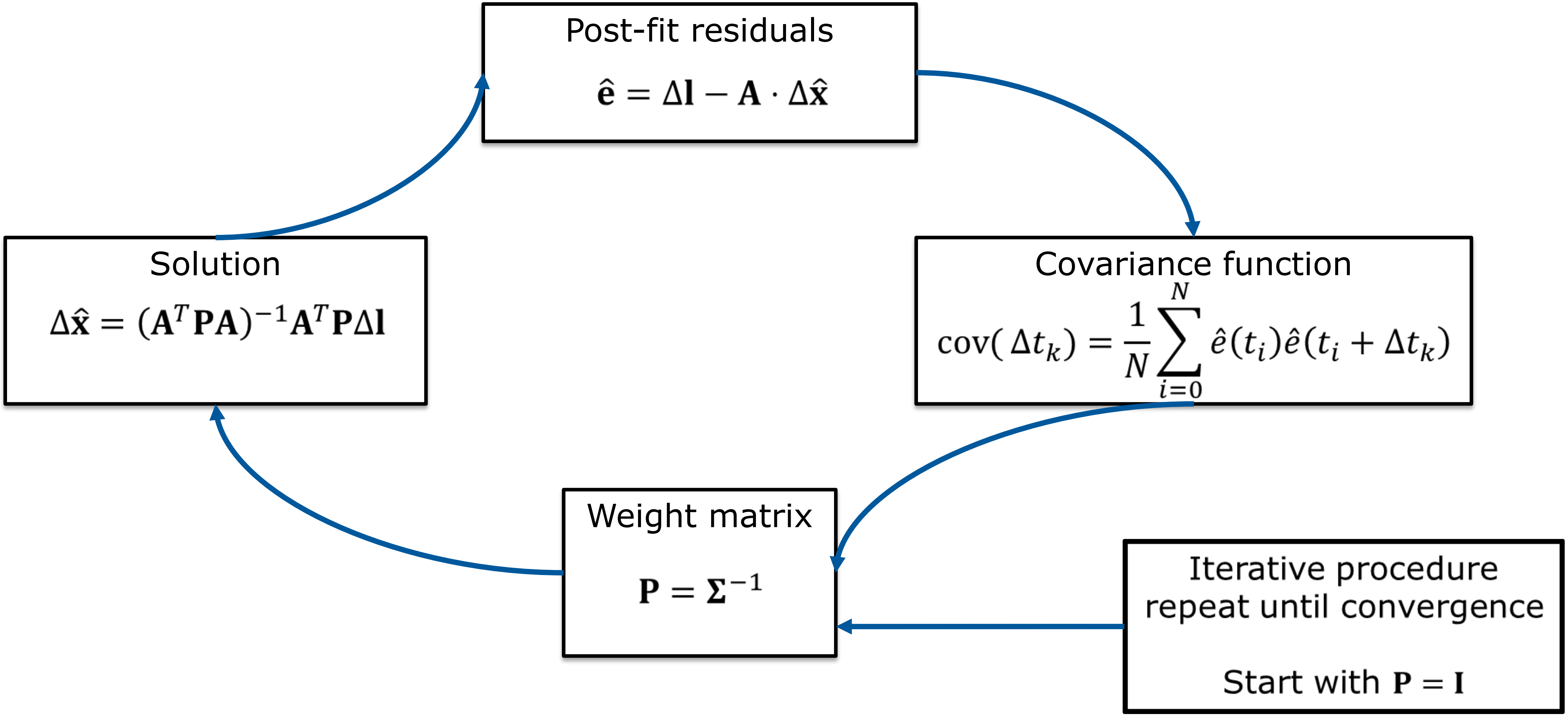
$$\boldsymbol{\Sigma} = \begin{pmatrix} \mathrm{cov}(\Delta t_0) & \mathrm{cov}(\Delta t_1) & \mathrm{cov}(\Delta t_2) & \\ \mathrm{cov}(\Delta t_1) & \mathrm{cov}(\Delta t_0) & \mathrm{cov}(\Delta t_1) & \dots \\ \mathrm{cov}(\Delta t_2) & \mathrm{cov}(\Delta t_1) & \mathrm{cov}(\Delta t_0) & \\ & \vdots & & \end{pmatrix}$$

- Weight matrix

$$\mathbf{P} = \boldsymbol{\Sigma}^{-1}$$



Weight matrix of instrument noise



Post-fit residuals

$$\hat{\mathbf{e}} = \Delta\mathbf{l} - \mathbf{A}\cdot\Delta\hat{\mathbf{x}}$$

Covariance function

$$\mathrm{cov}(\Delta t_k) = \frac{1}{N}\sum_{i=0}^{N}\hat{e}(t_i)\hat{e}(t_i+\Delta t_k)$$

Weight matrix

$$\mathbf{P} = \boldsymbol{\Sigma}^{-1}$$

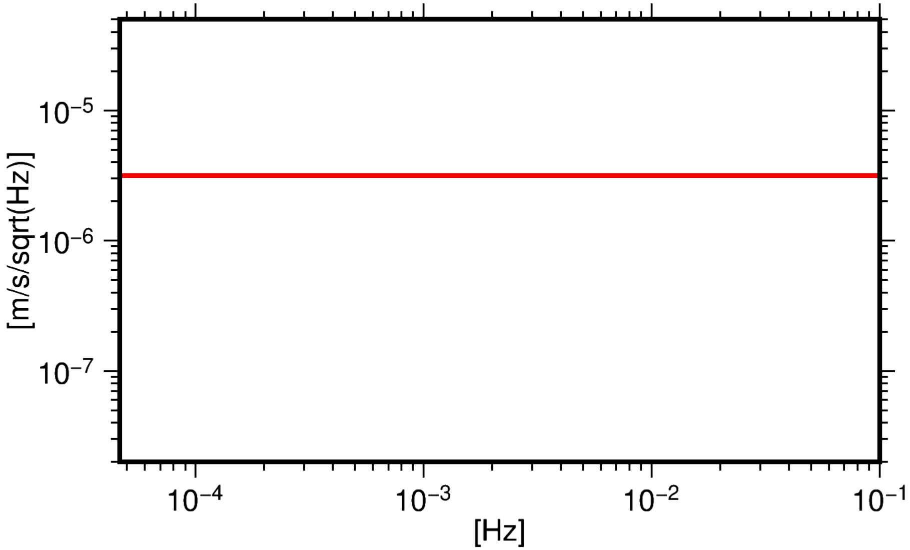
Solution

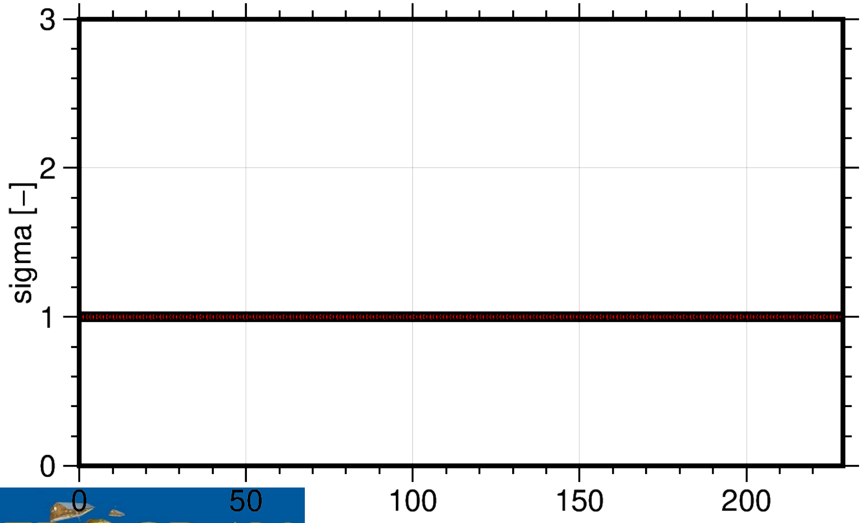
$$\Delta\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\Delta\mathbf{l}$$

Iterative procedure repeat until convergence

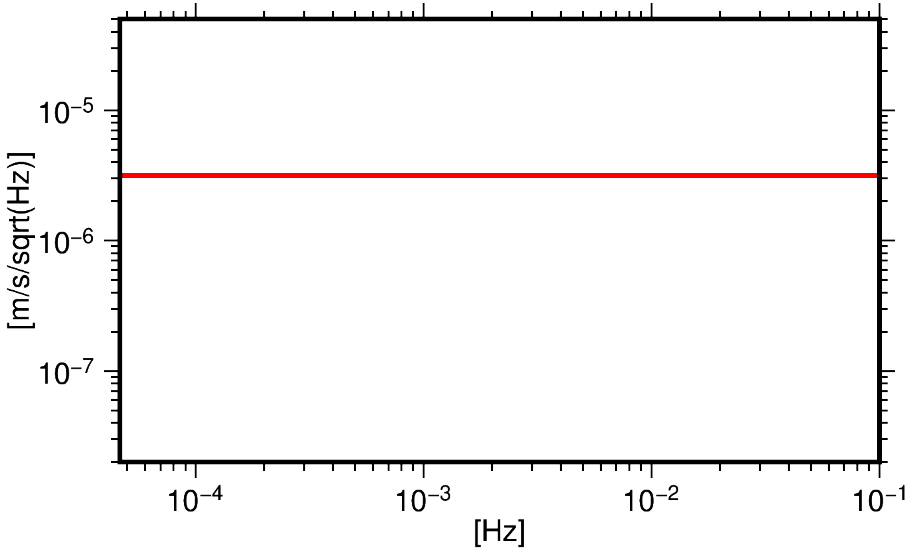
Start with $\mathbf{P} = \mathbf{I}$

0. iteration


PSD range rate residuals (2010–02)


arc sigmas (2010–02)

0. iteration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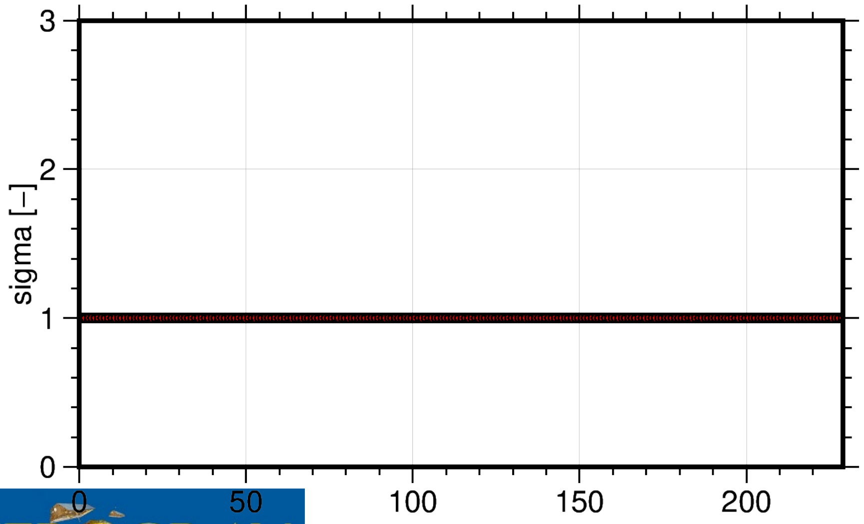
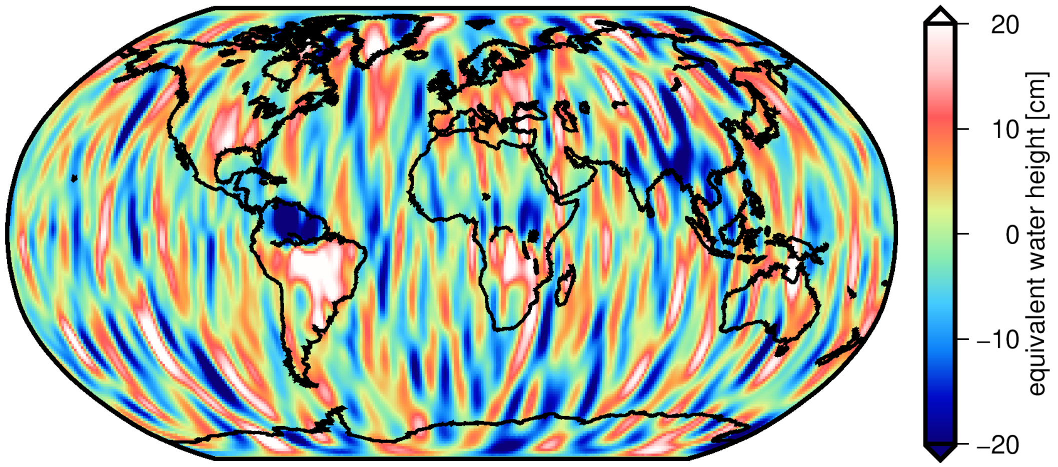
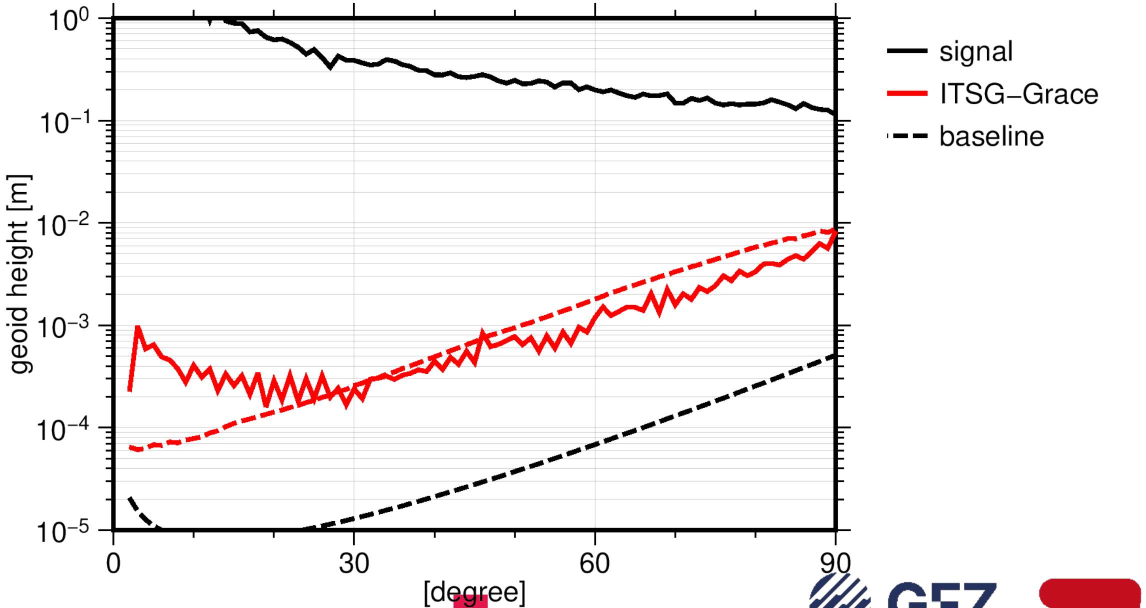

arc sigmas (2010–02)

First weight matrix

Least squares adjustment

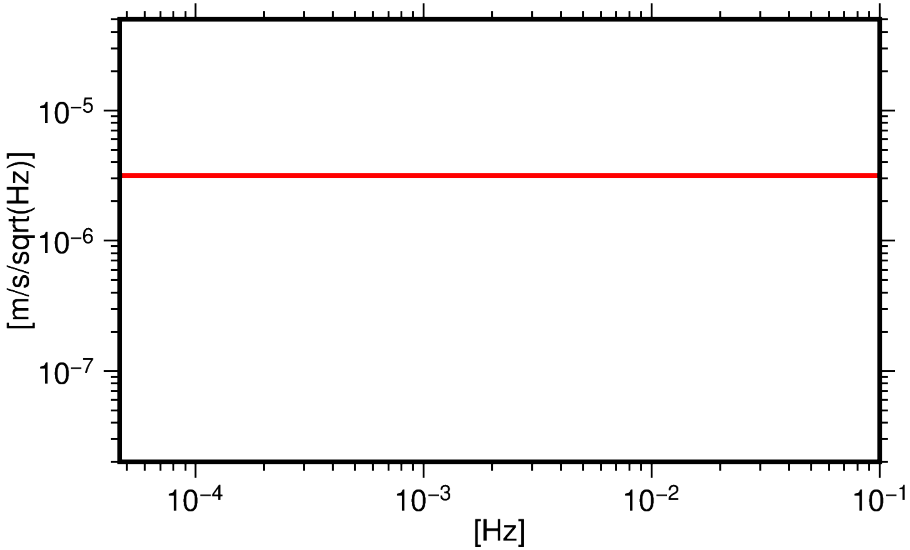
Gaussian filter 350 km



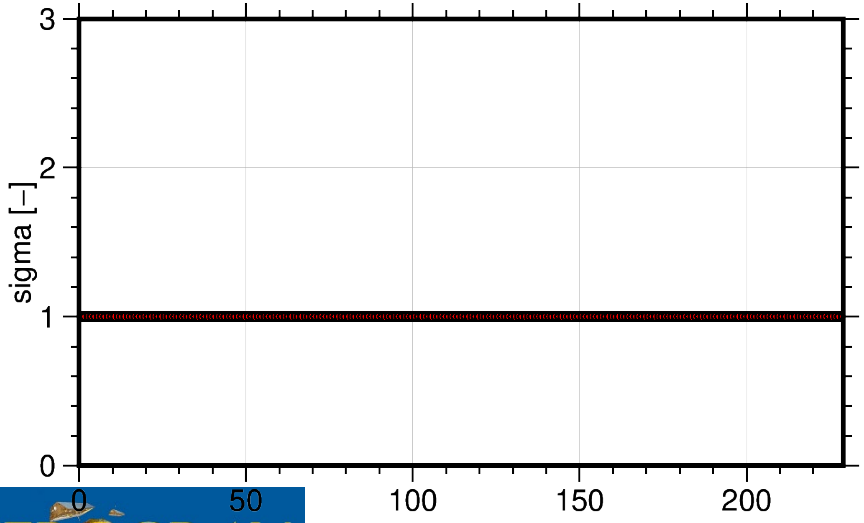

degree amplitudes (2010–02)

0. iteration

PSD range rate residuals (2010–02)



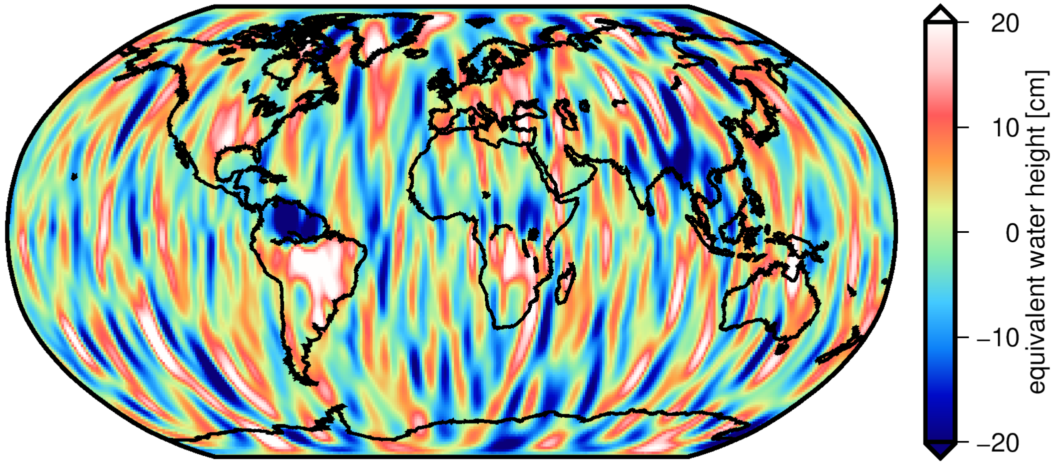
arc sigmas (2010–02)



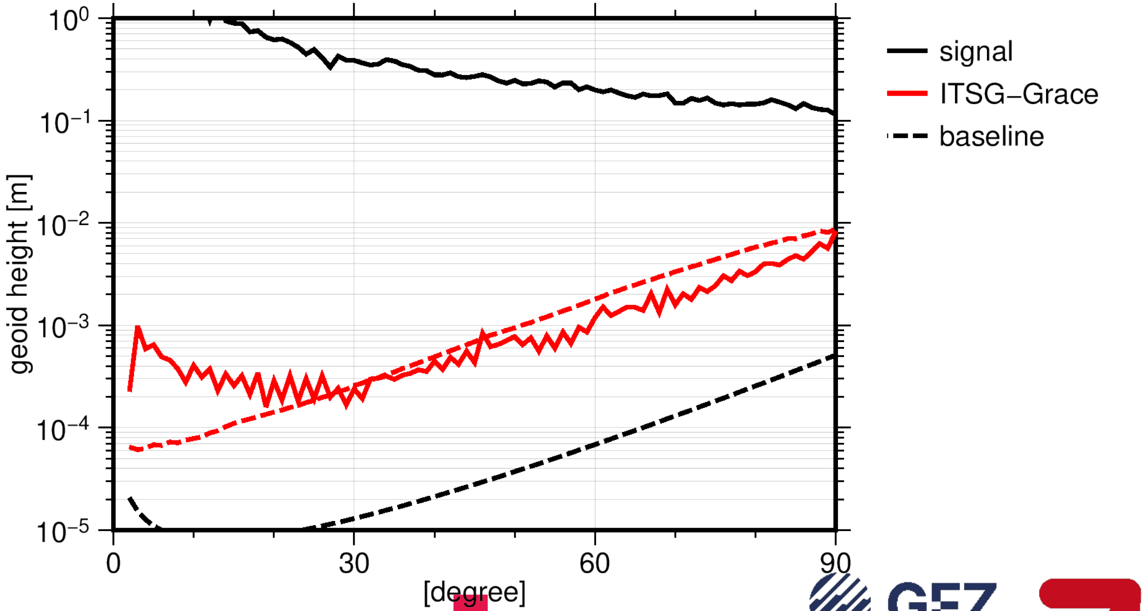
Post-fit residuals

Estimate new covariance matrix

Gaussian filter 350 km



degree amplitudes (2010–02)



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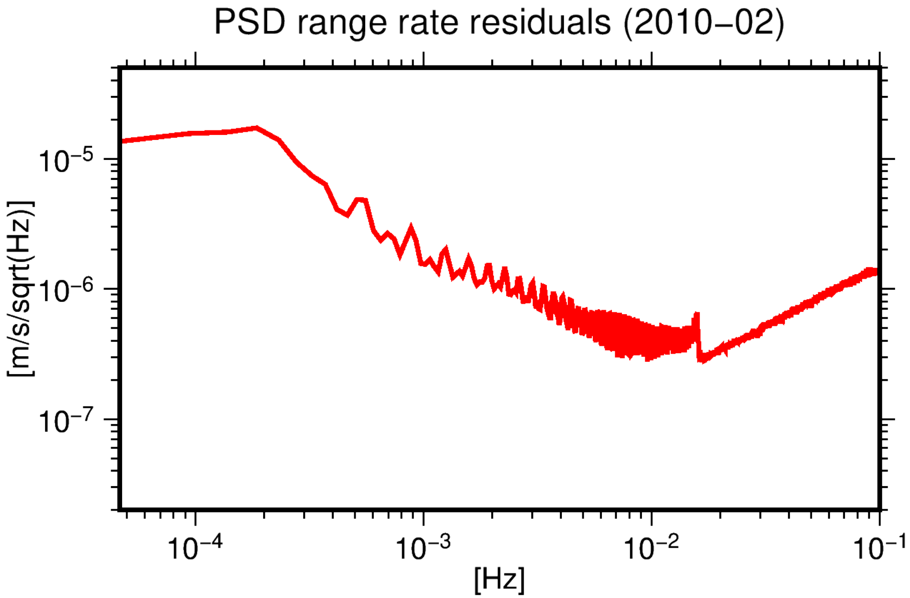
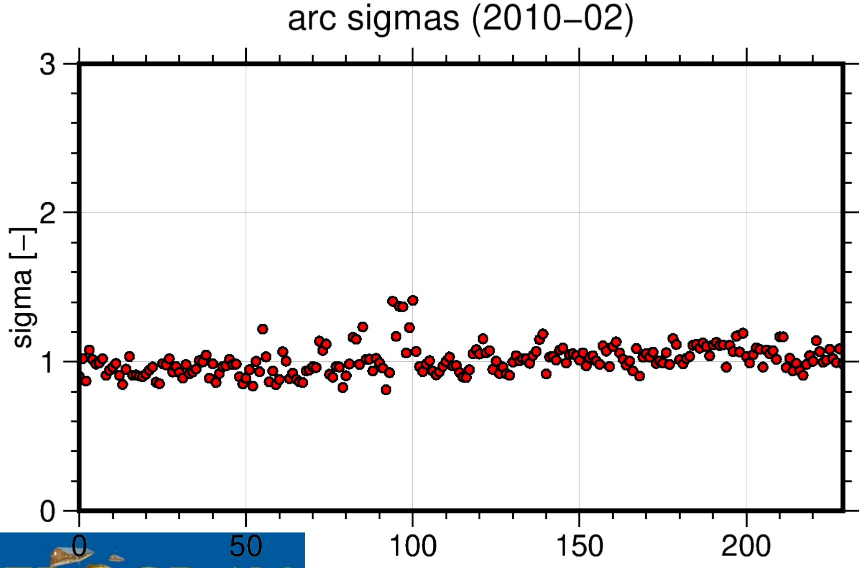
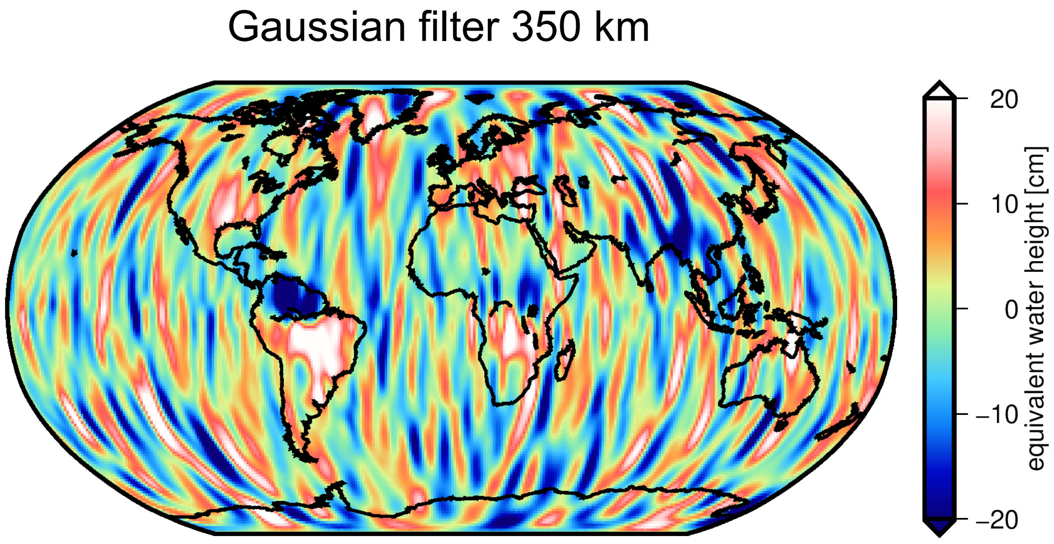
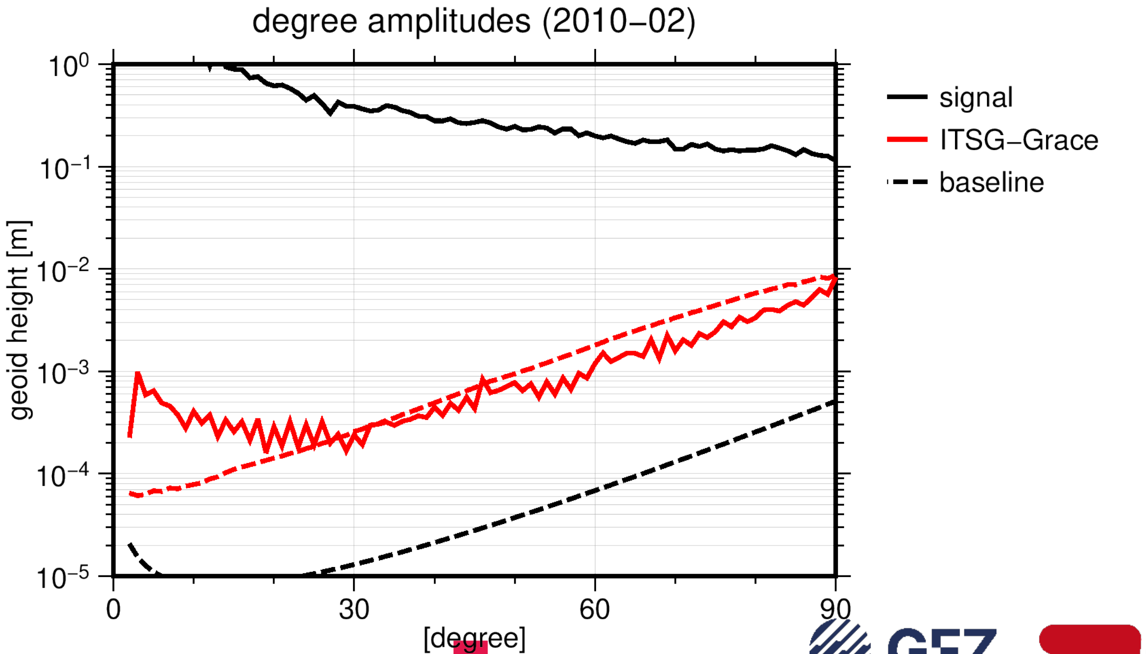




Post-fit residuals

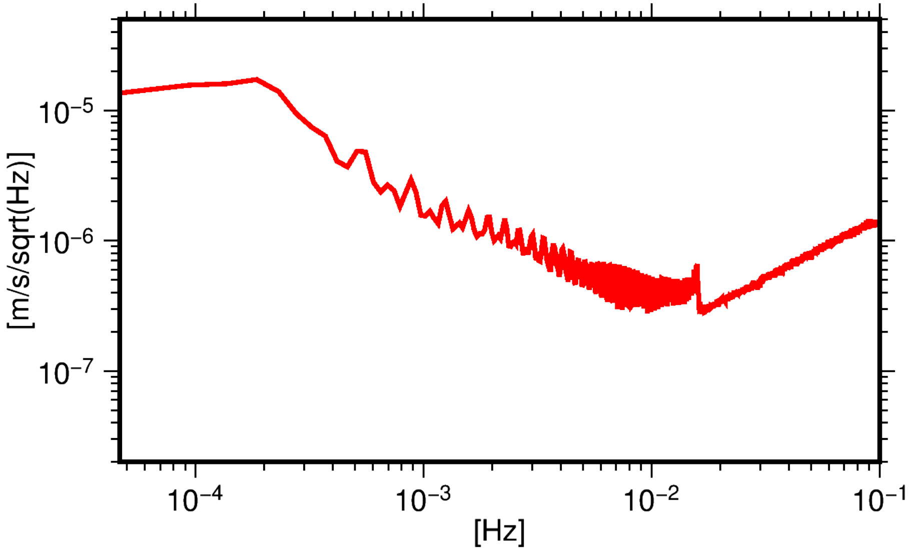
Estimate new covariance matrix



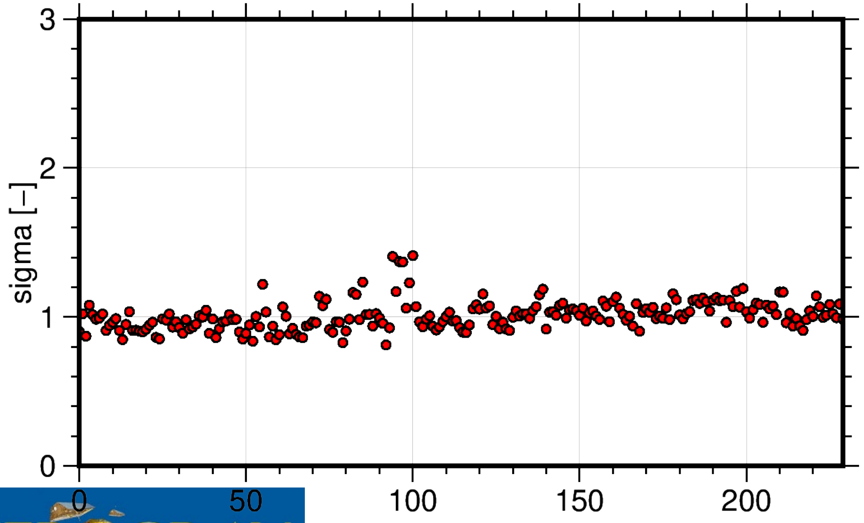


0. iteration

PSD range rate residuals (2010–02)



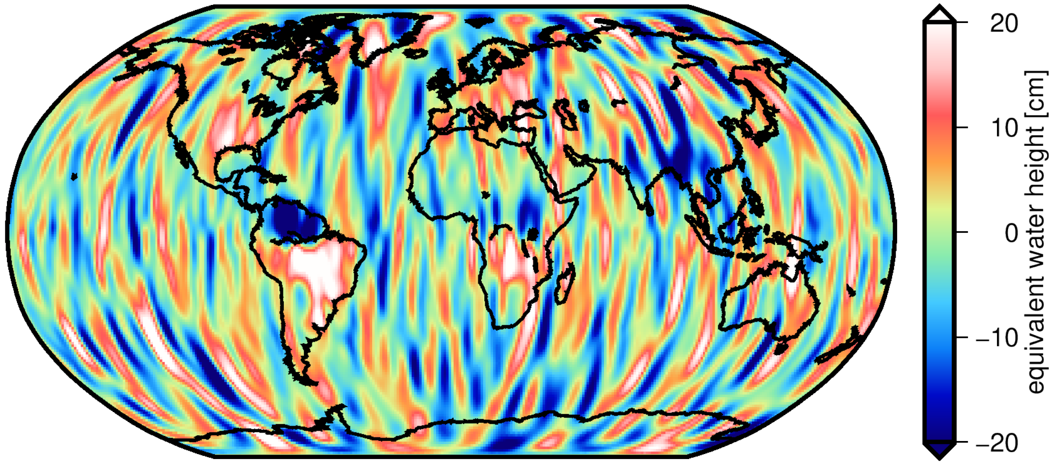
arc sigmas (2010–02)



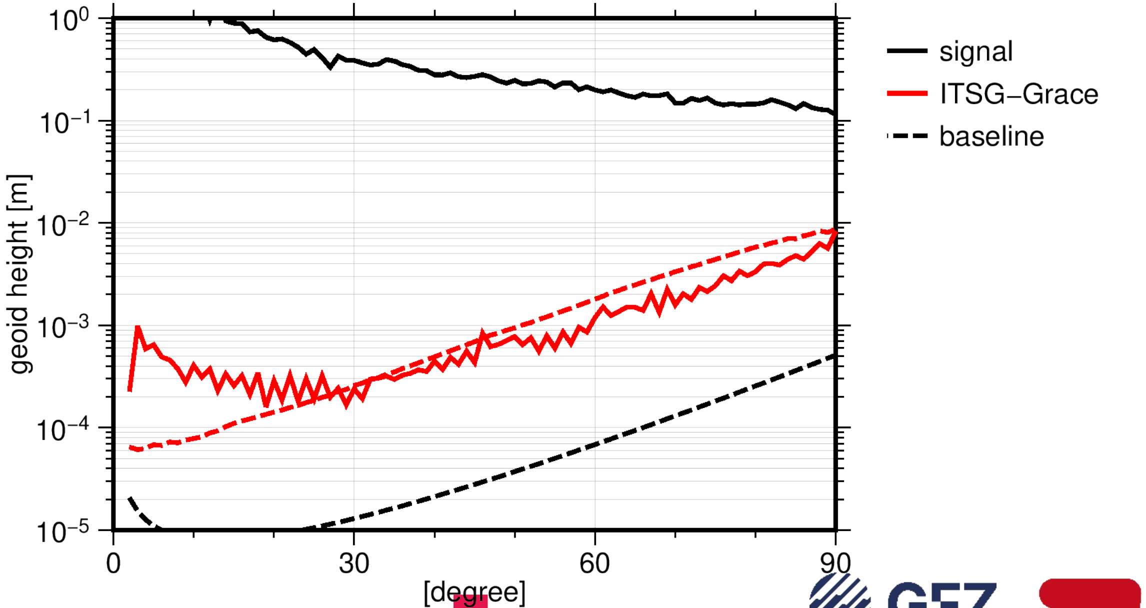
New weight matrix

Least squares adjustment

Gaussian filter 350 km

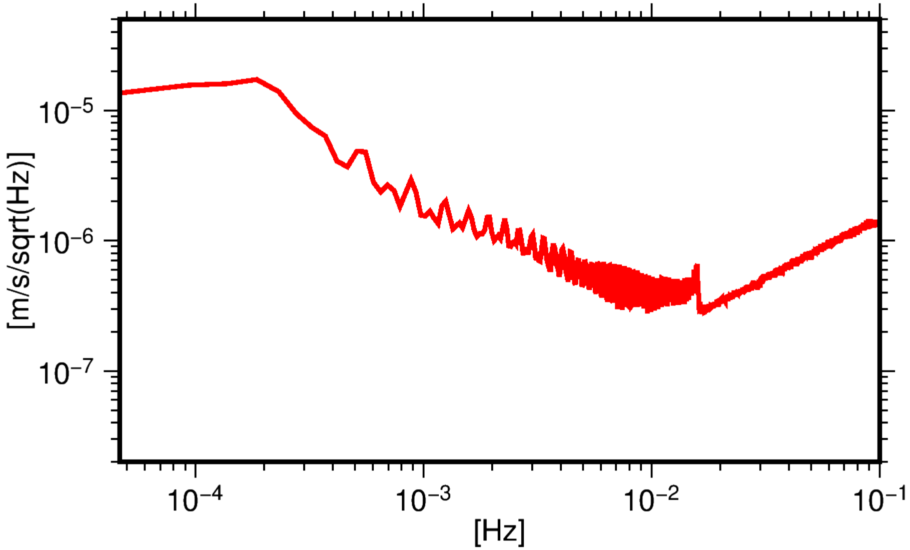


degree amplitudes (2010–02)

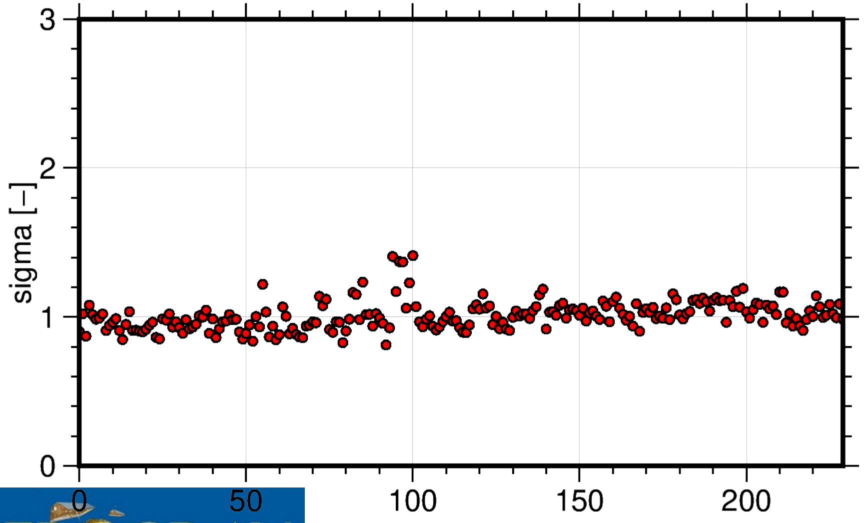


1. iteration

PSD range rate residuals (2010–02)



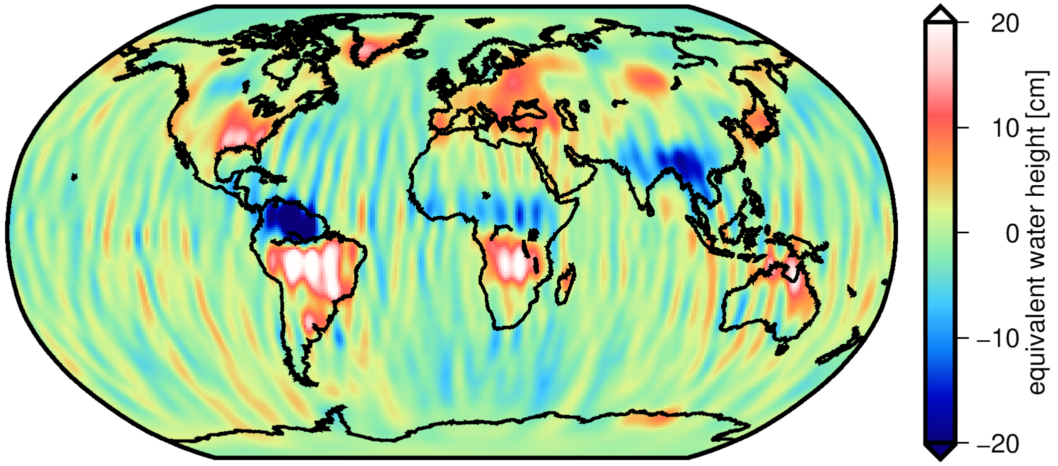
arc sigmas (2010–02)



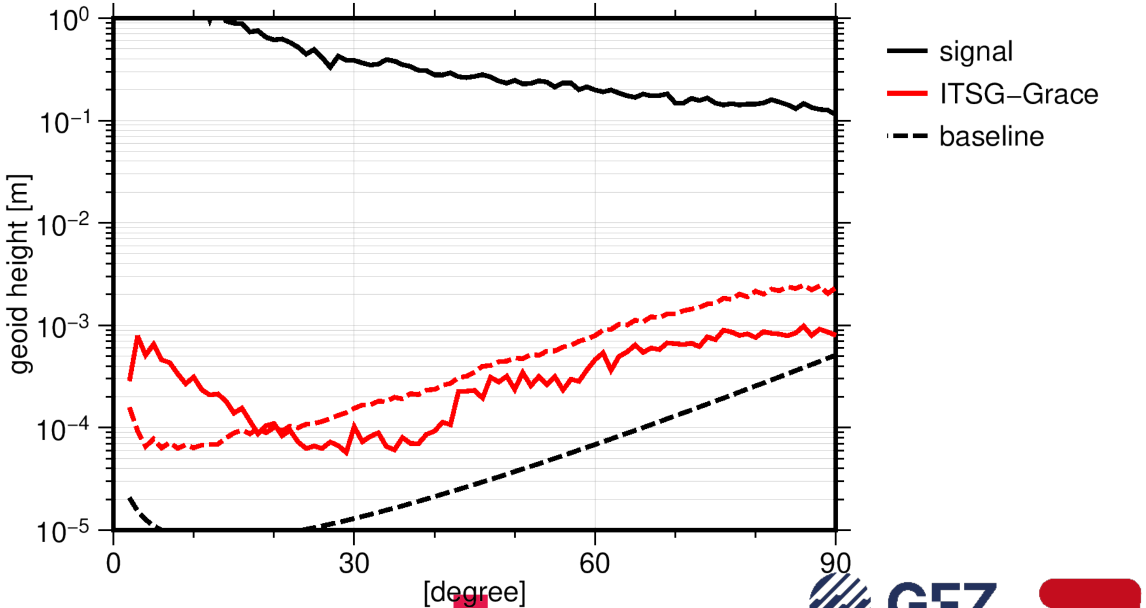
New weight matrix

Least squares adjustment

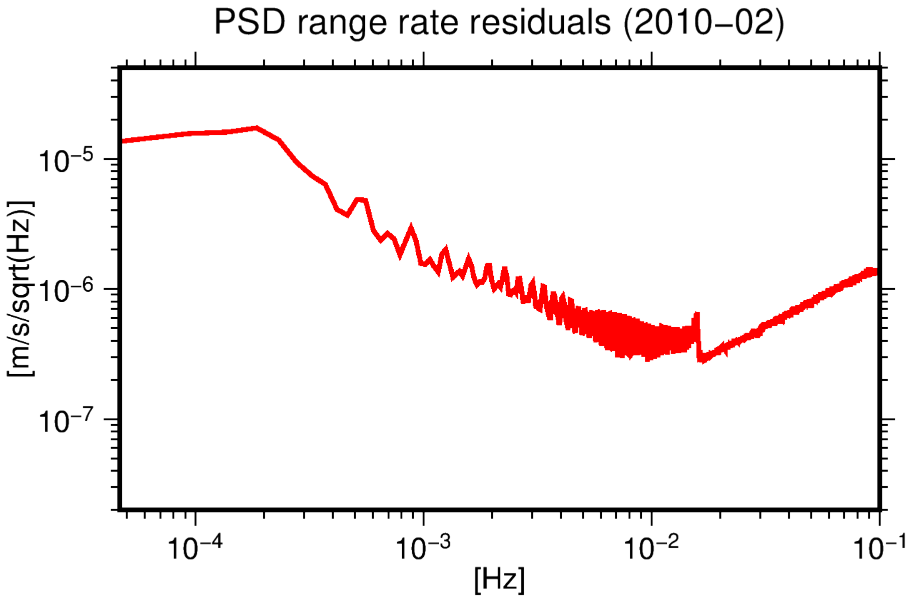
Gaussian filter 350 km

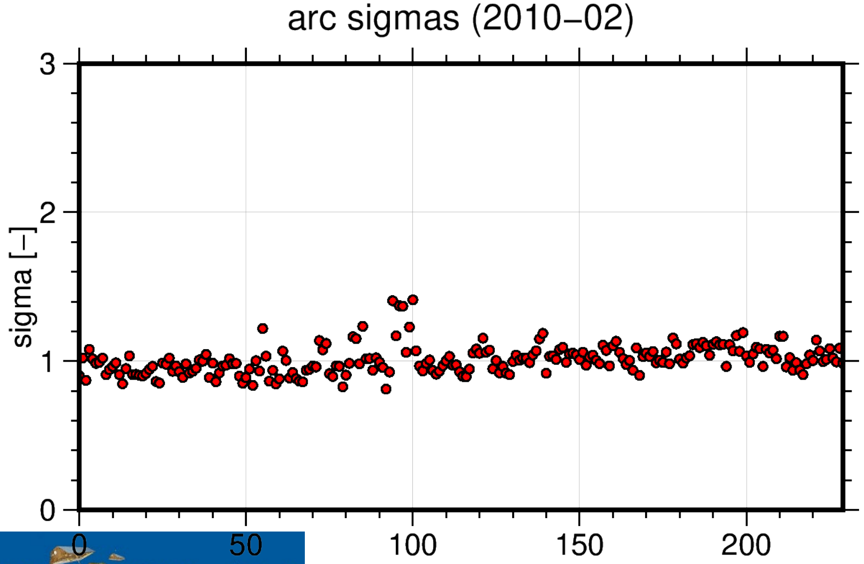


degree amplitudes (2010–02)



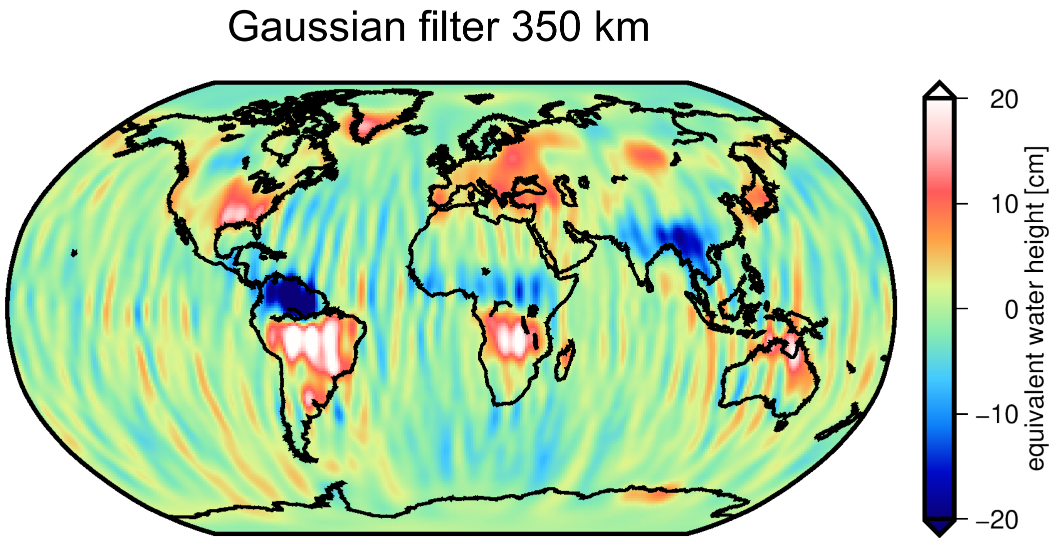
1. iteration

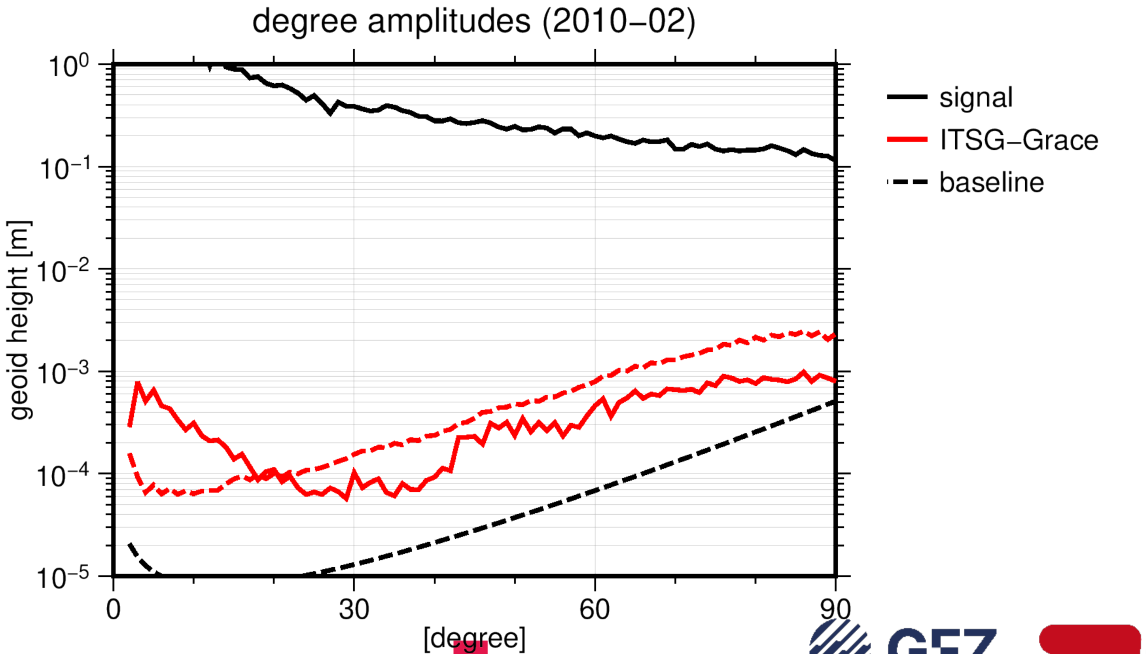




Post-fit residuals

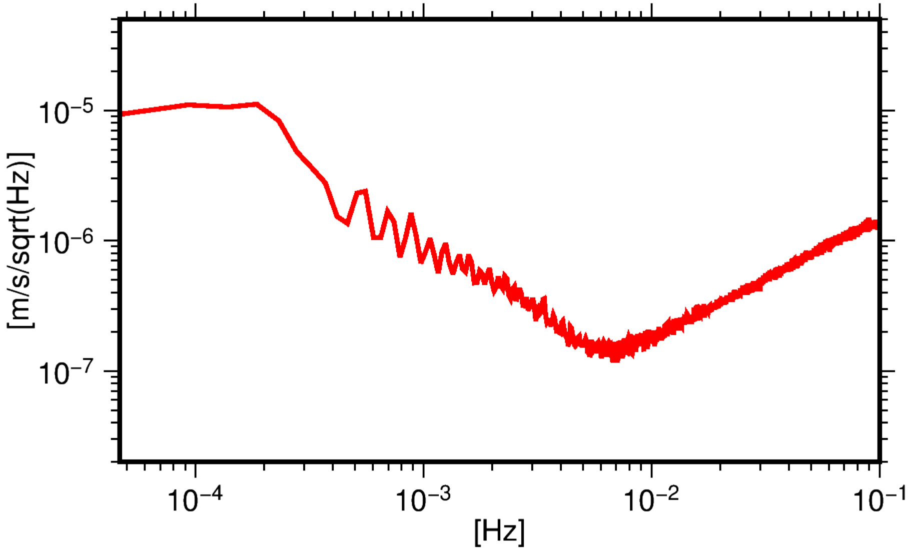
Estimate new covariance matrix



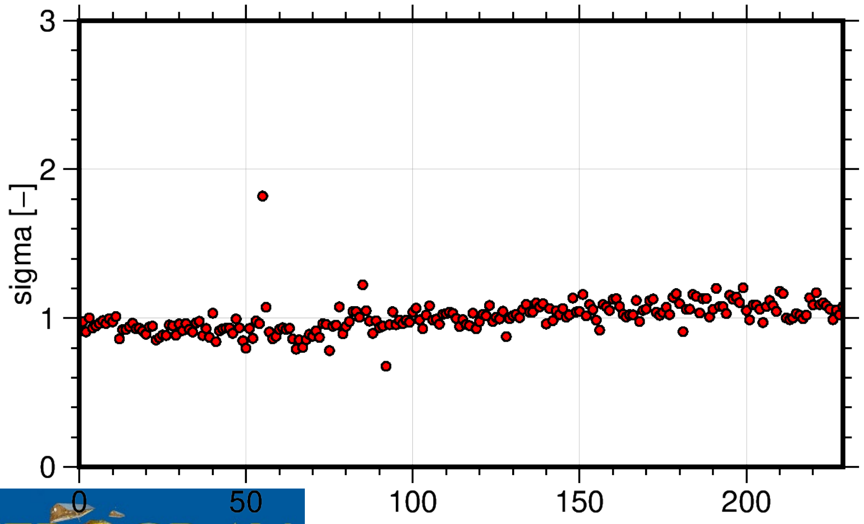


1. iteration

PSD range rate residuals (2010–02)



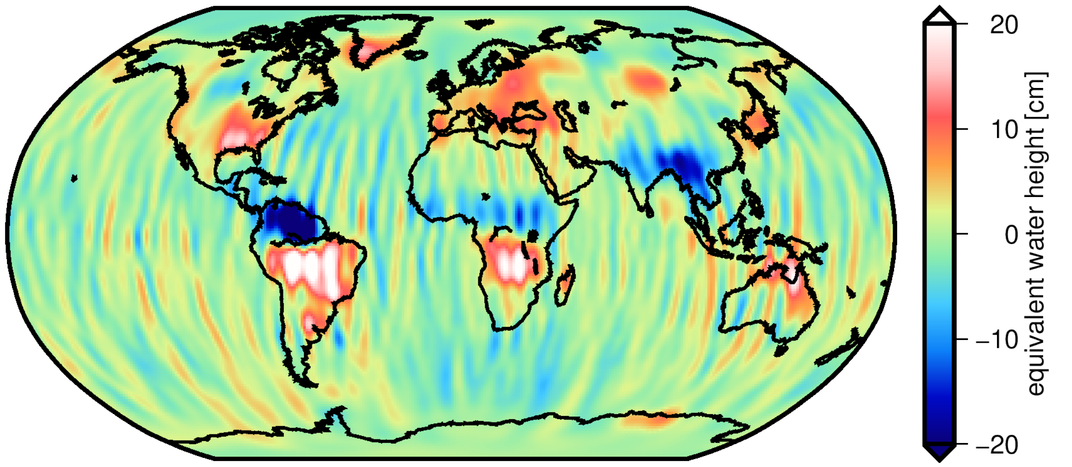
arc sigmas (2010–02)



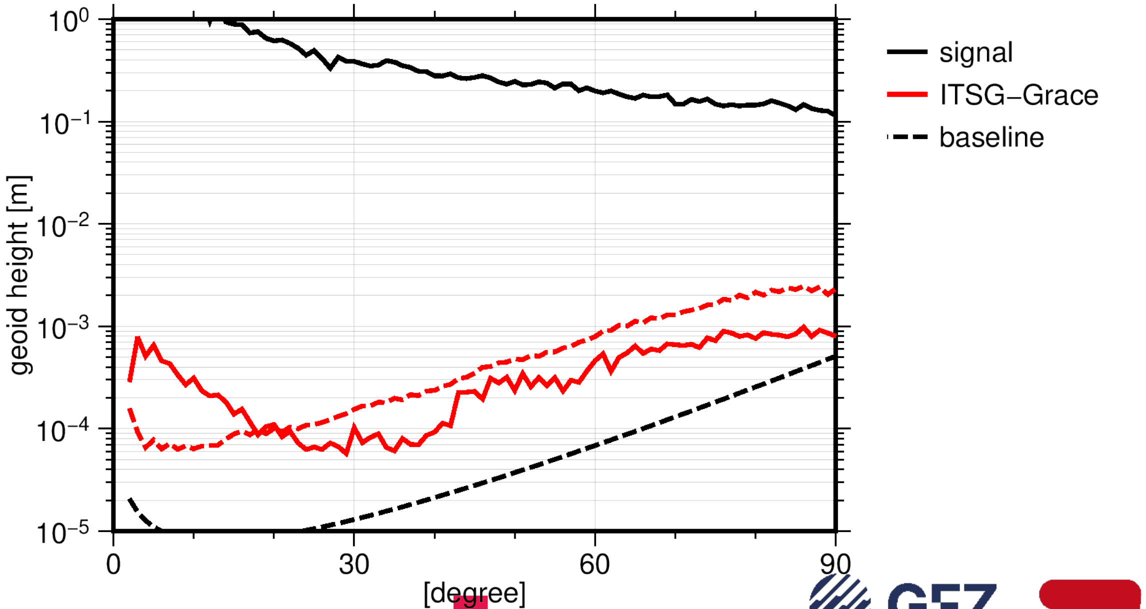
Post-fit residuals

Estimate new covariance matrix

Gaussian filter 350 km

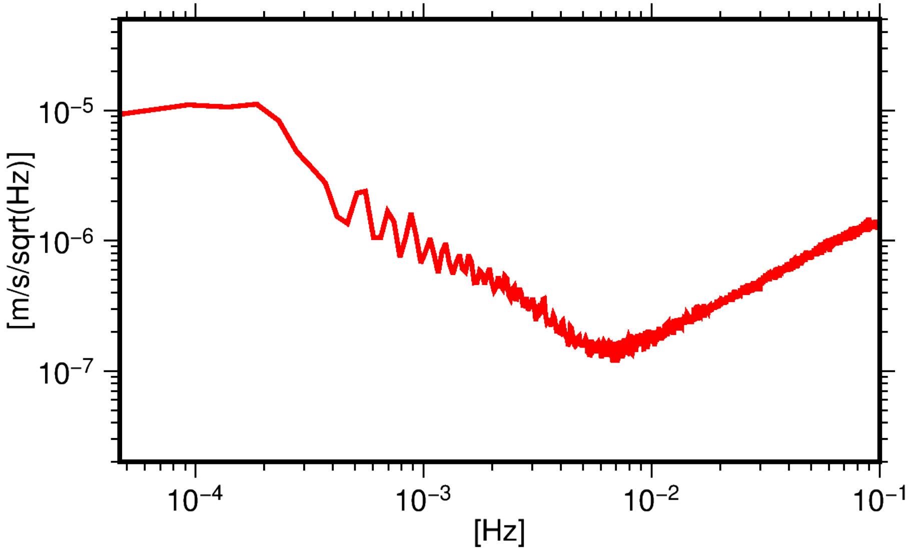


degree amplitudes (2010–02)

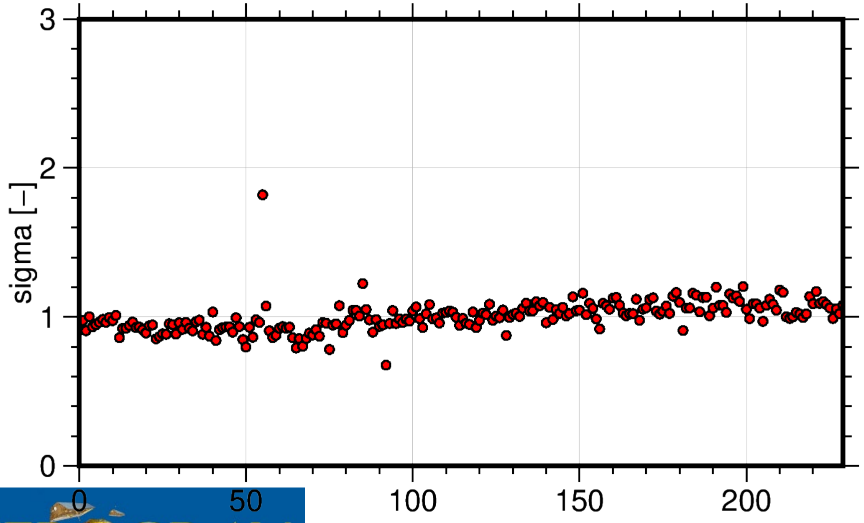


1. iteration

PSD range rate residuals (2010–02)



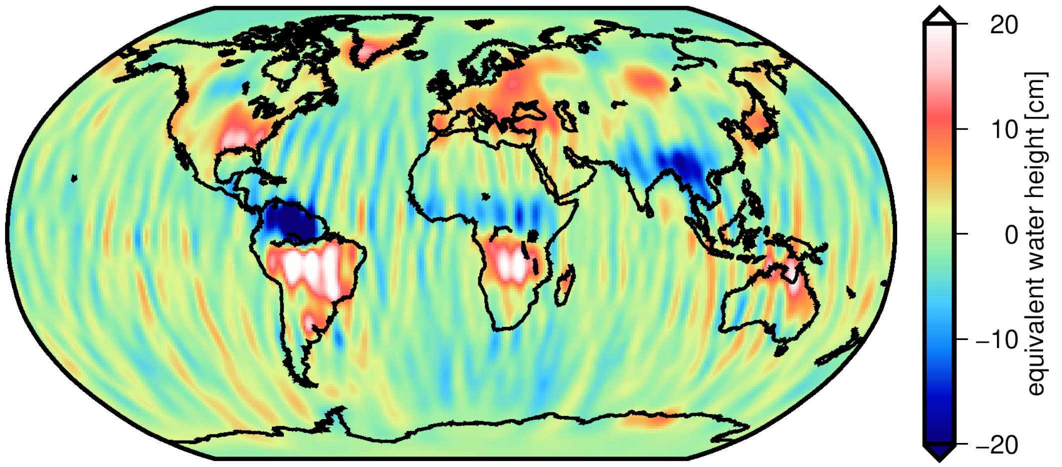
arc sigmas (2010–02)



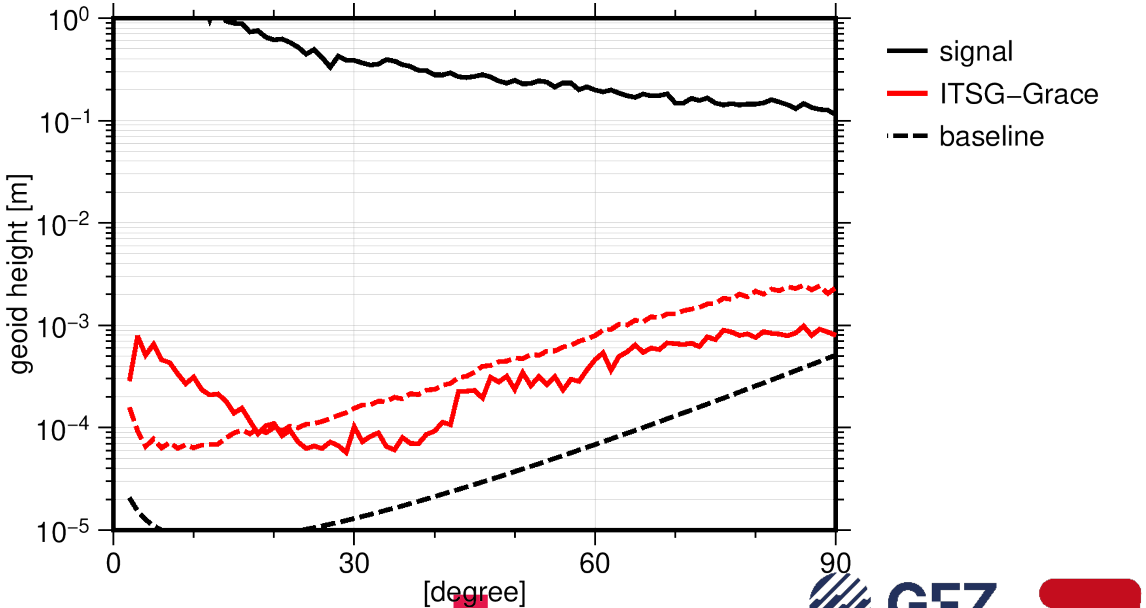
New weight matrix

Least squares adjustment

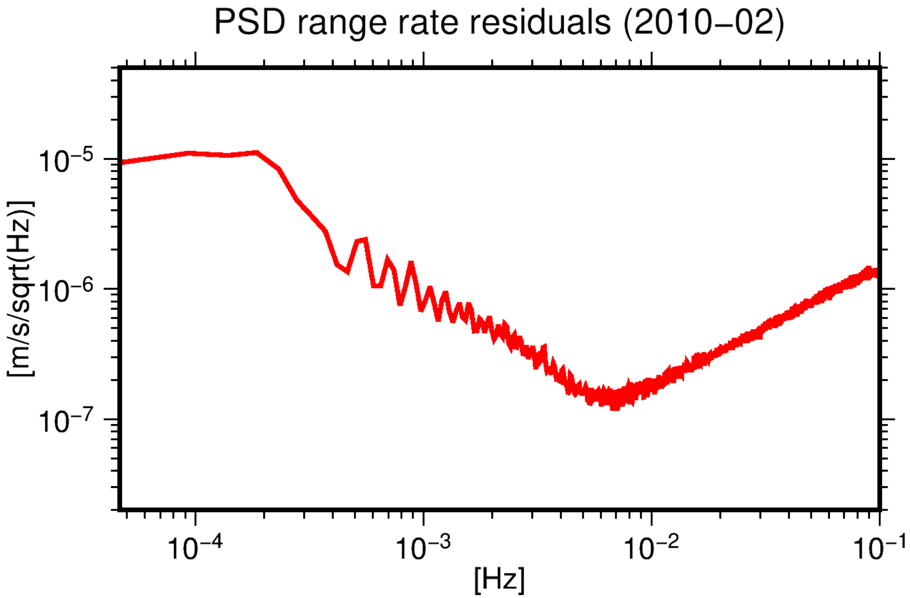
Gaussian filter 350 km

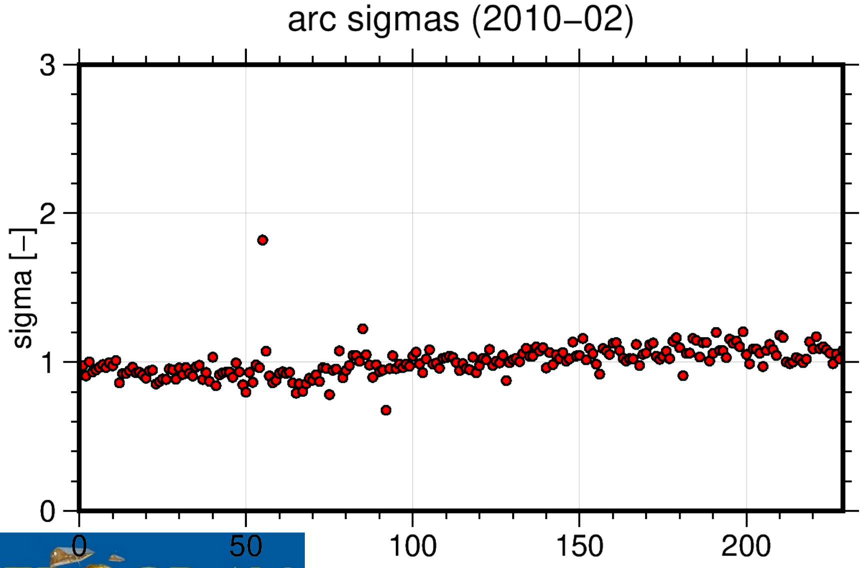


degree amplitudes (2010–02)



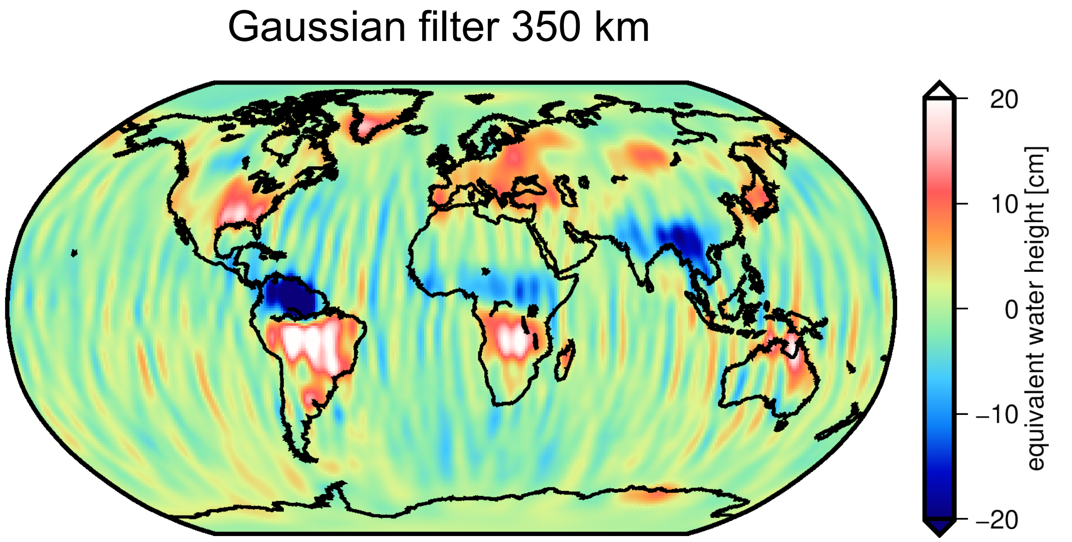
2. iteration

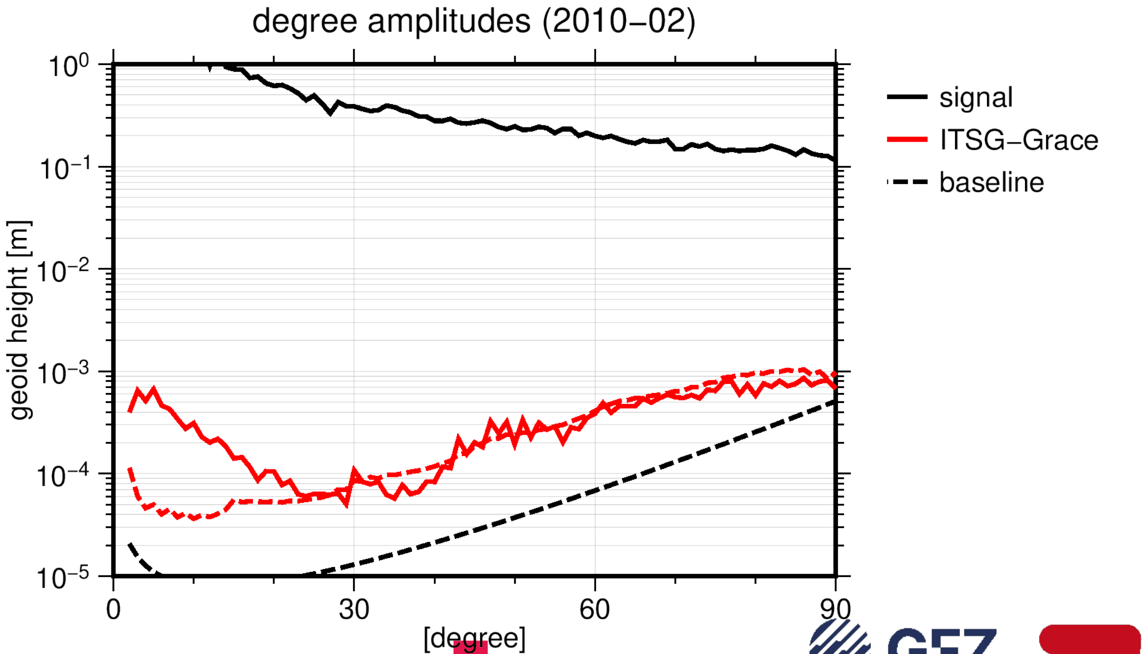




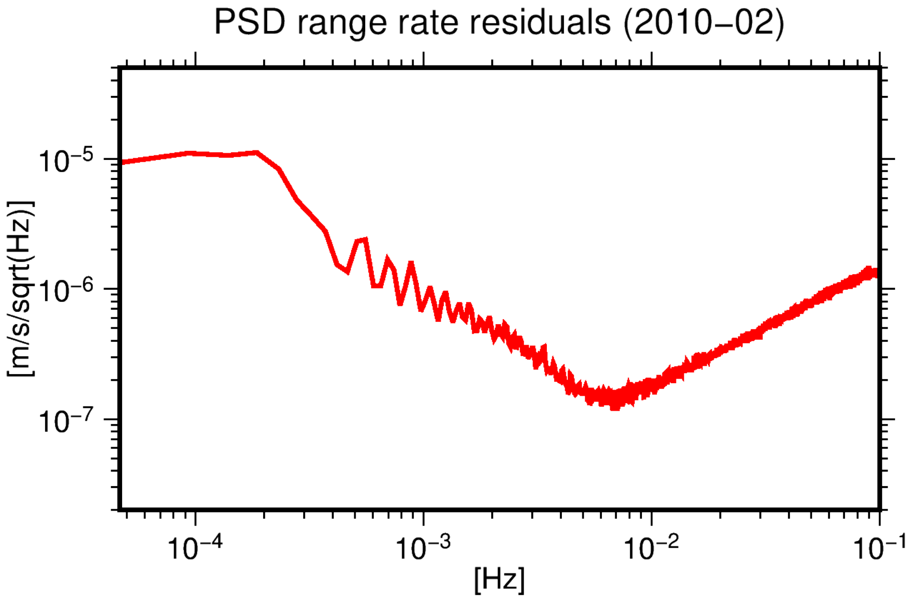
New weight matrix

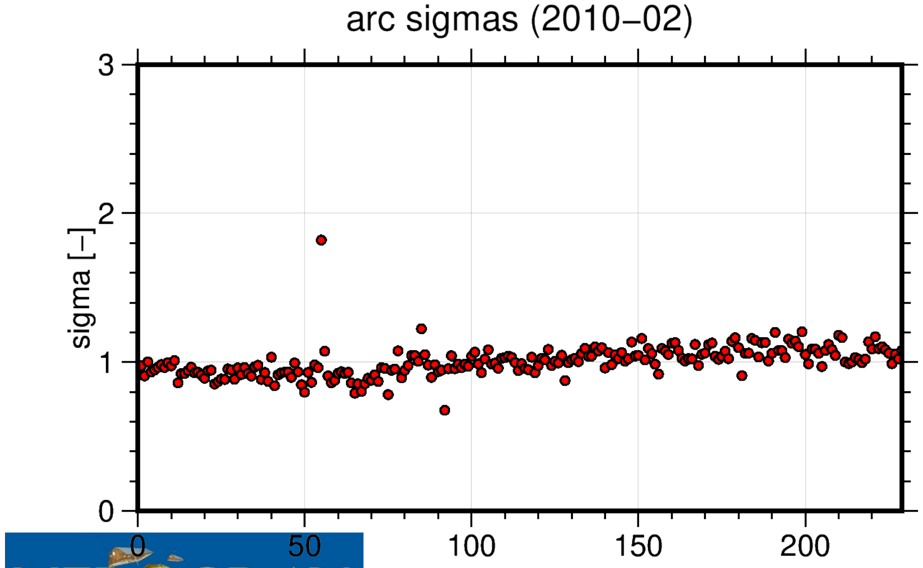
Least squares adjustment





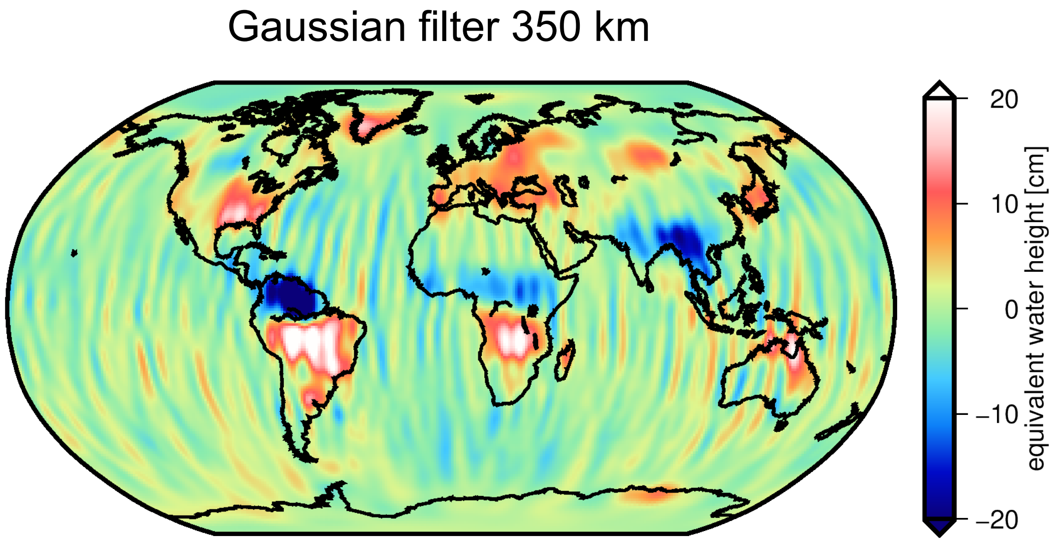
2. iteration

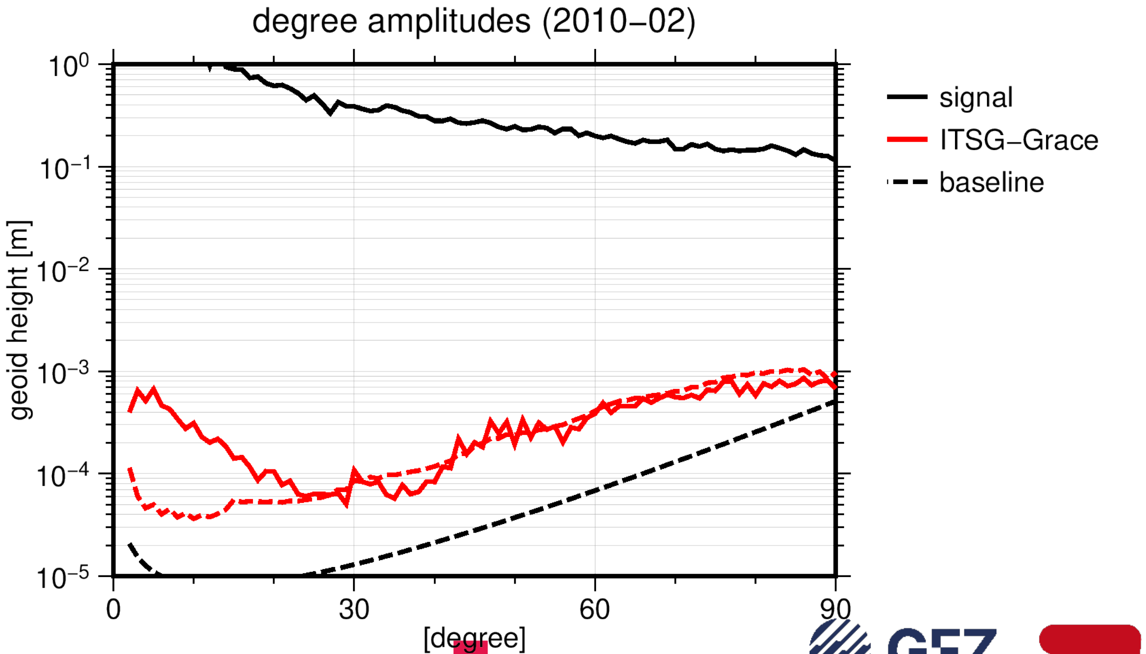

PSD range rate residuals (2010−02)


arc sigmas (2010−02)

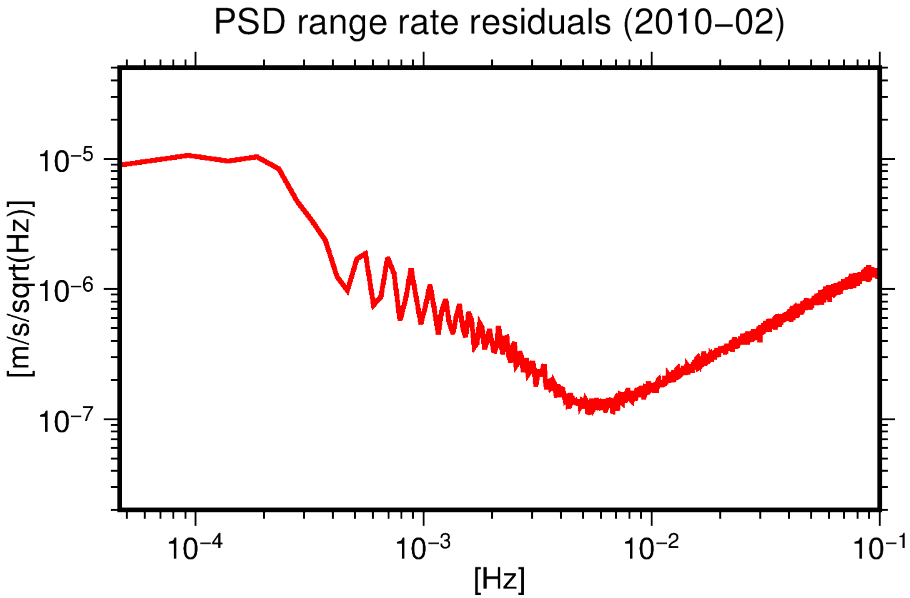
Post-fit residuals

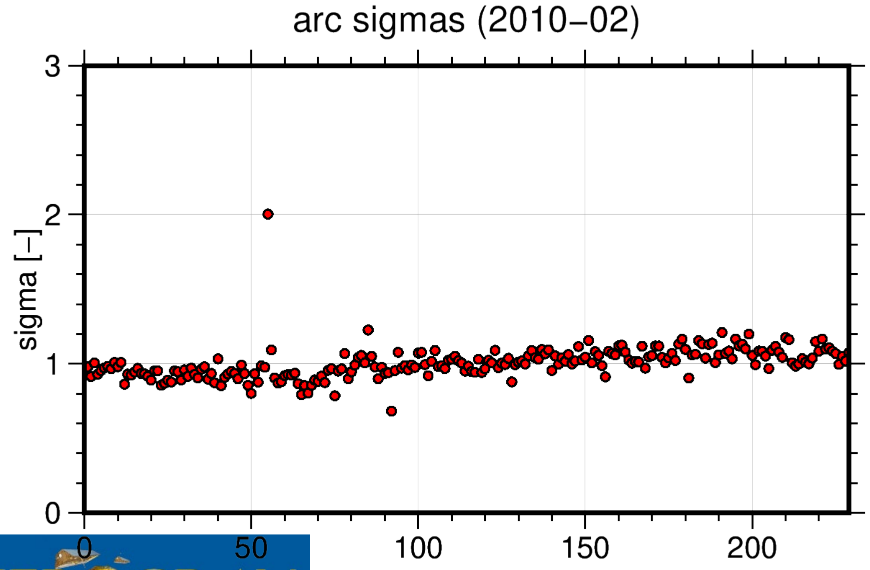
Estimate new covariance matrix


Gaussian filter 350 km


degree amplitudes (2010−02)

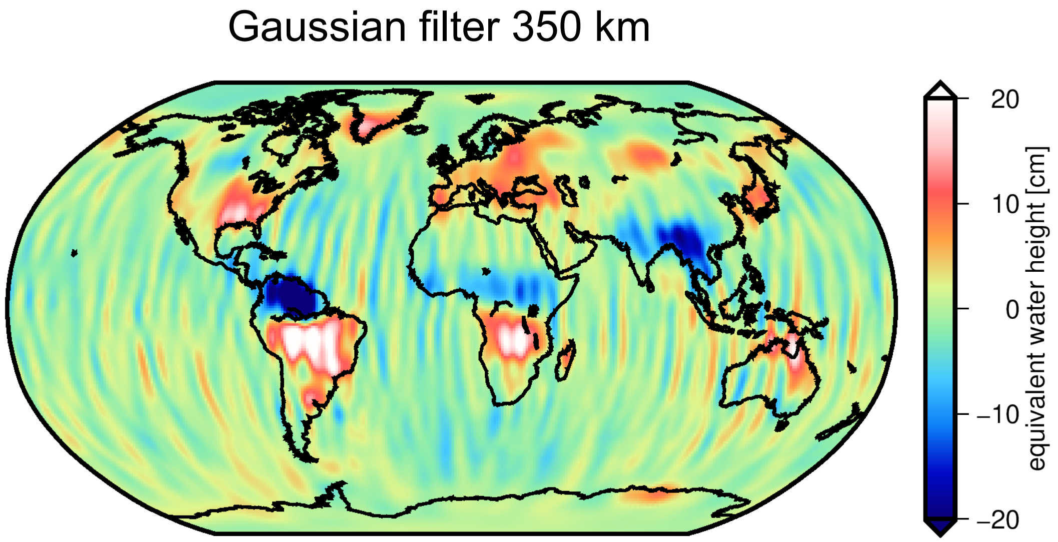
2. iteration

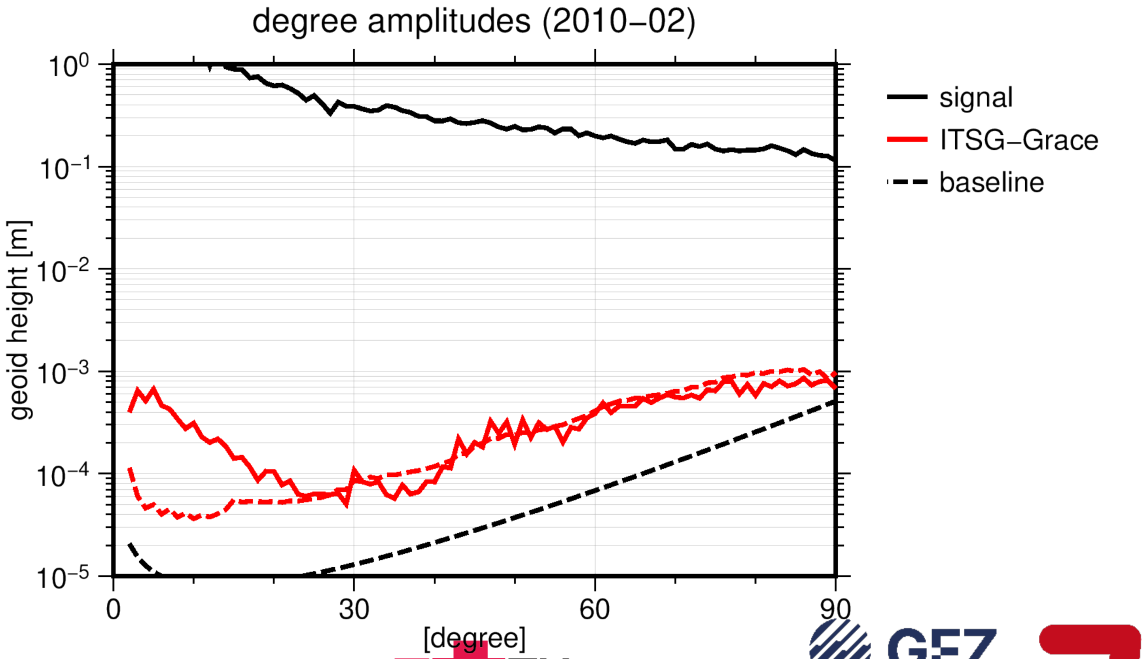




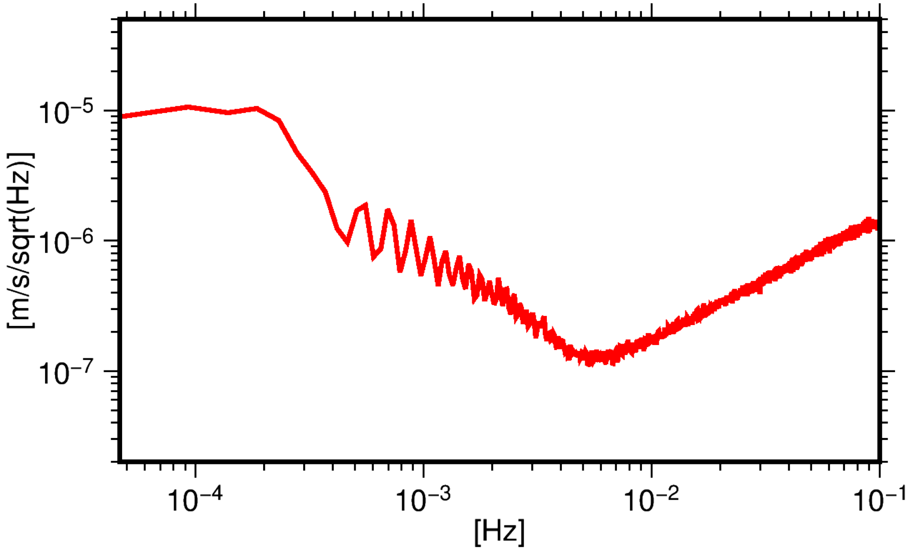
Post-fit residuals

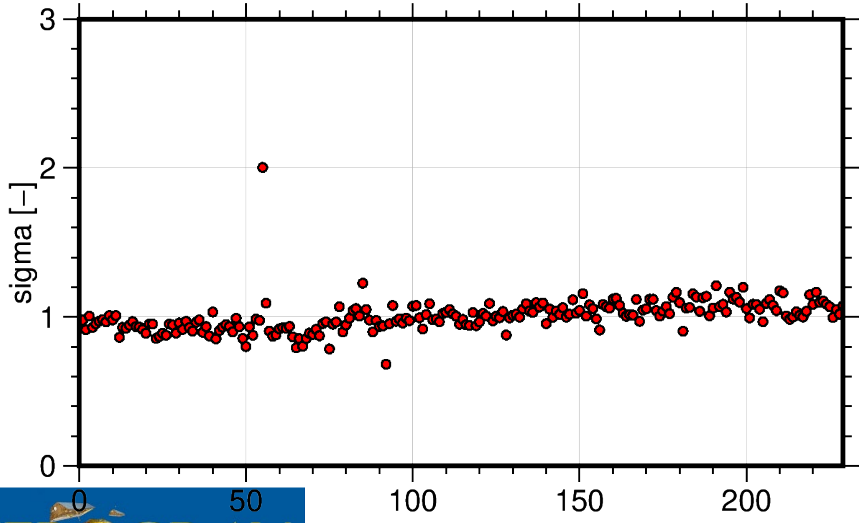
Estimate new covariance matrix





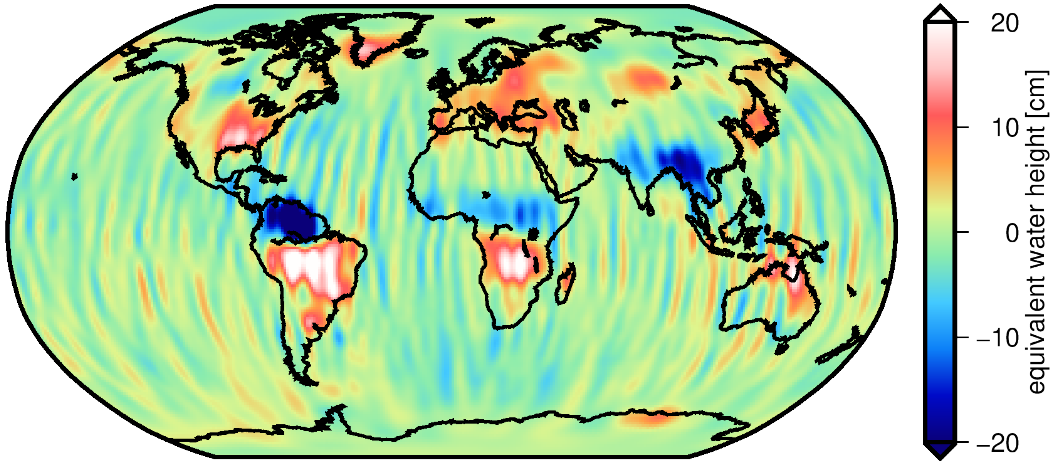
2. iteration

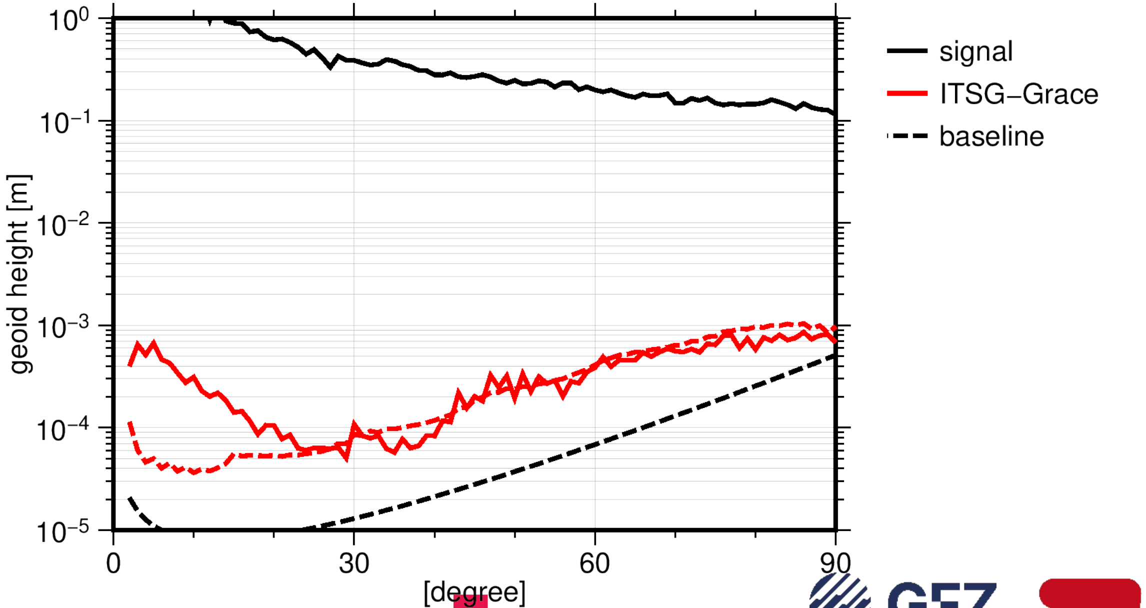

PSD range rate residuals (2010–02)


arc sigmas (2010–02)

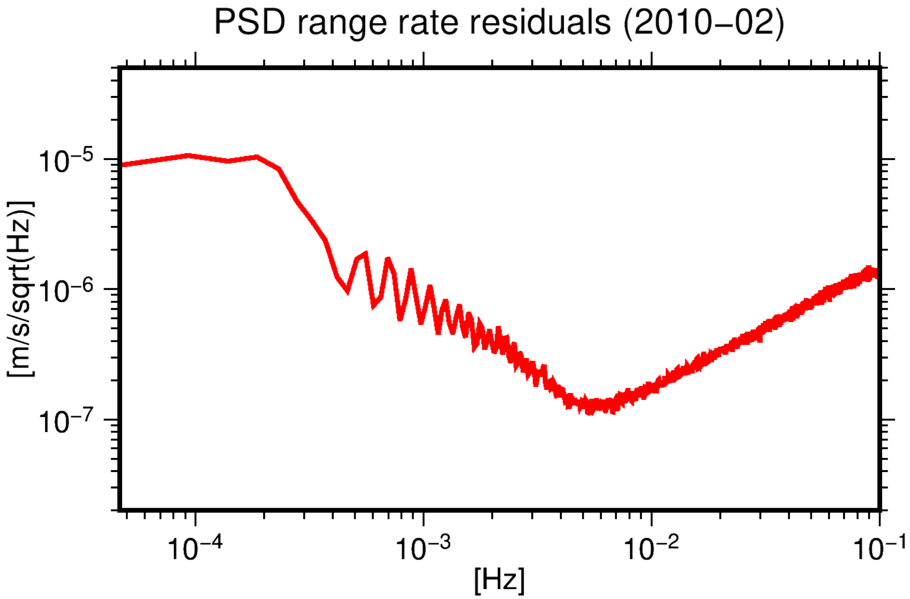
New weight matrix

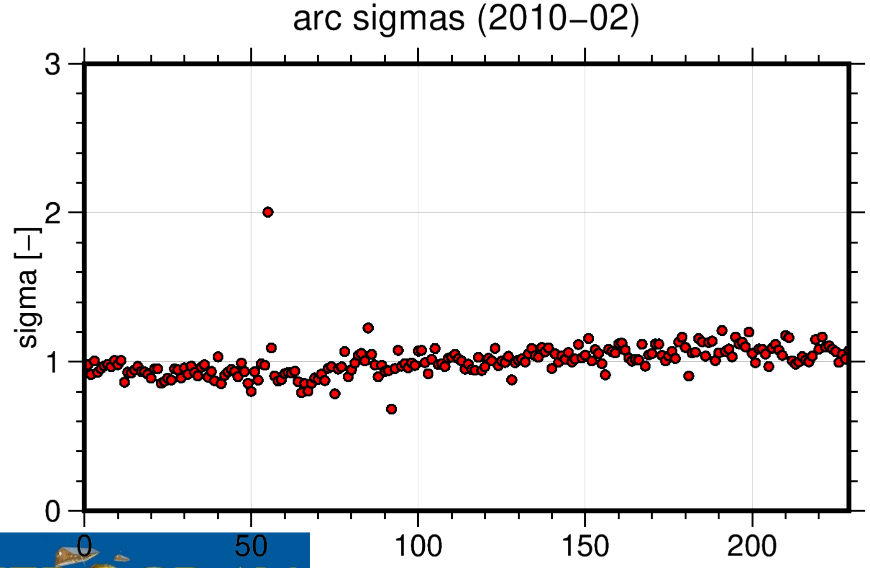
Least squares adjustment


Gaussian filter 350 km


degree amplitudes (2010–02)

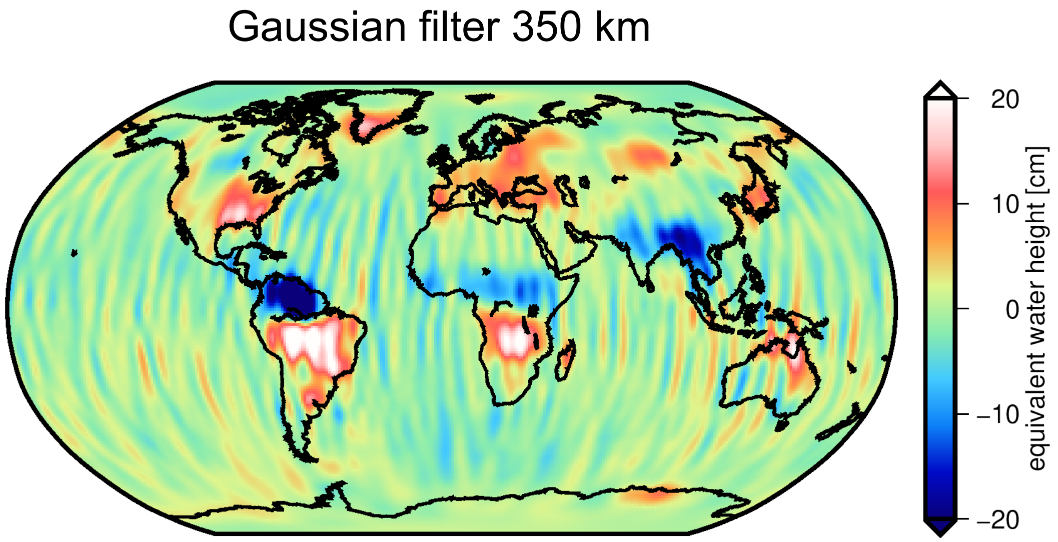
3. iteration

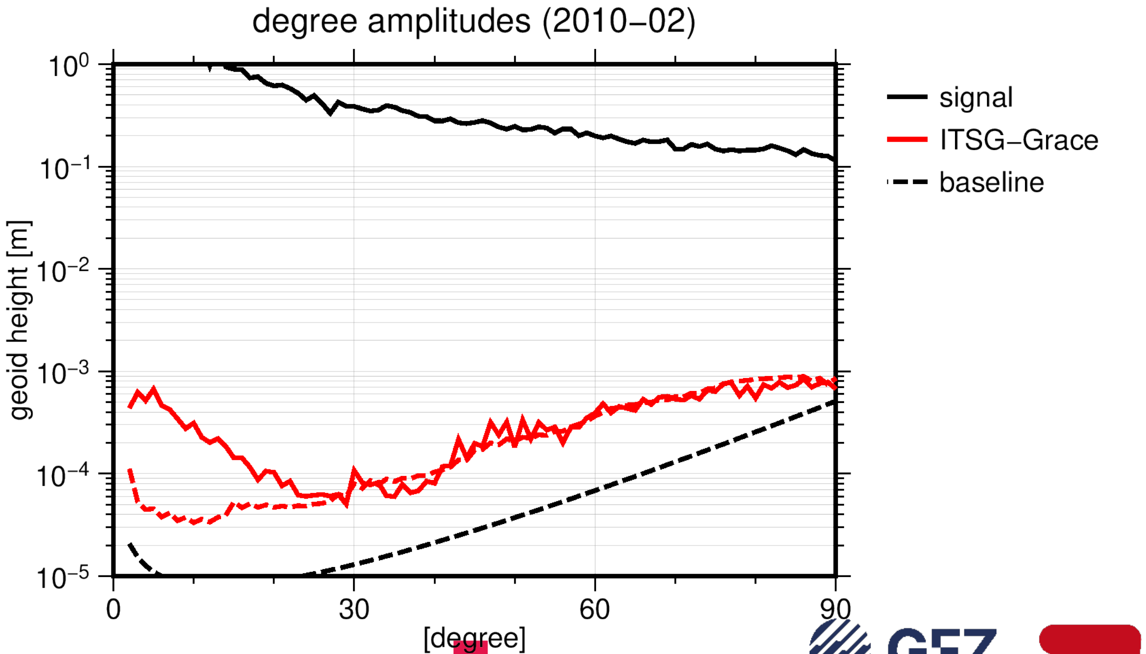




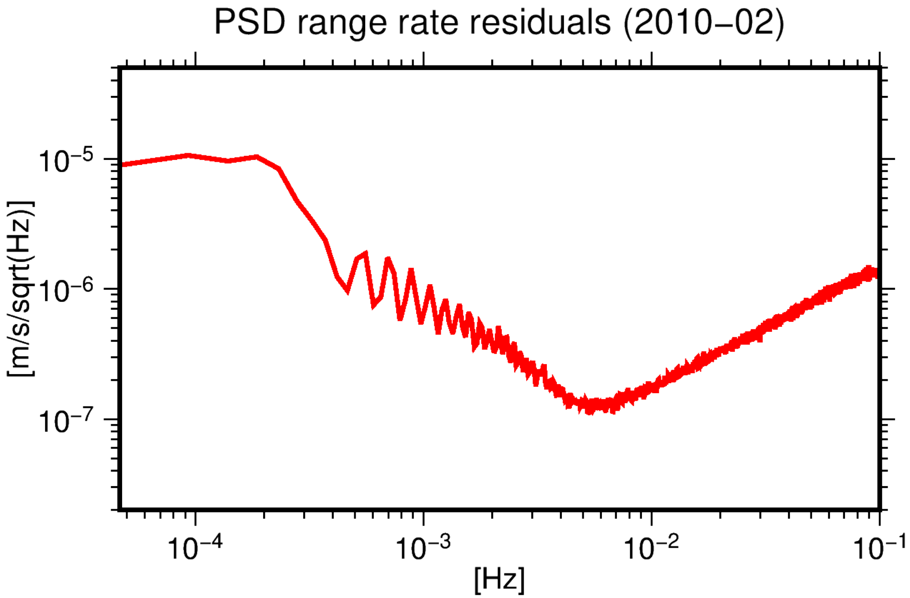
New weight matrix

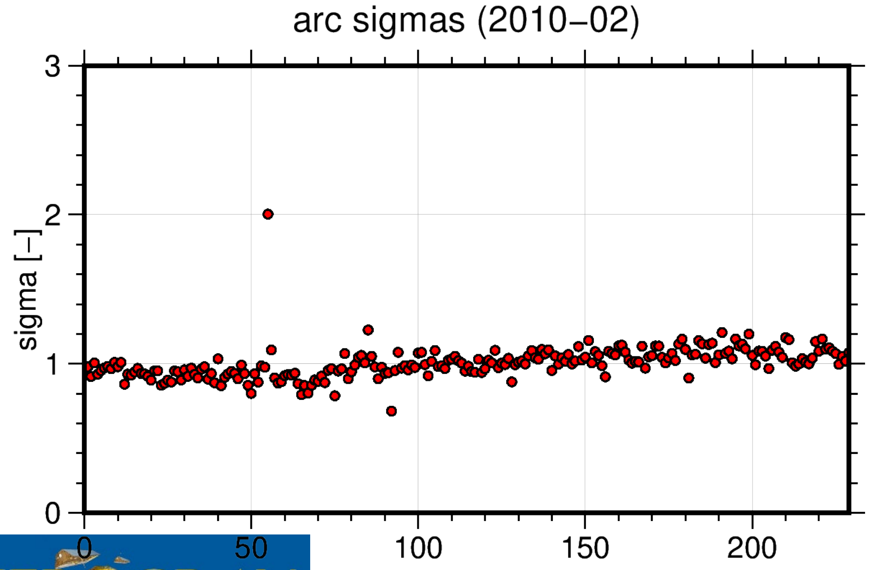
Least squares adjustment





3. iteration

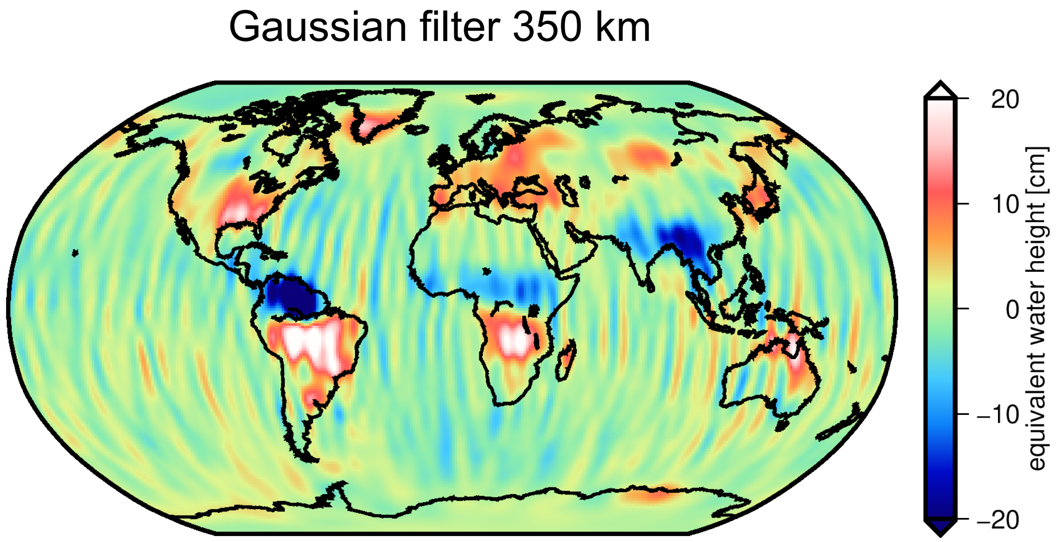

PSD range rate residuals (2010–02)

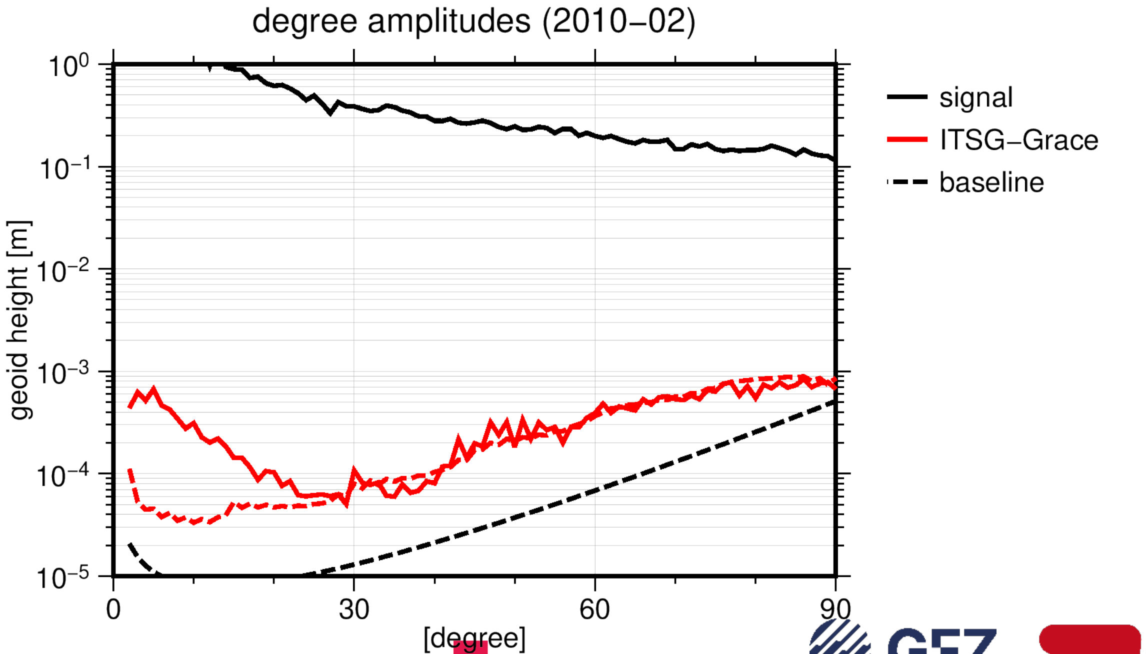

arc sigmas (2010–02)

Post-fit residuals

Estimate new covariance matrix

No more changes

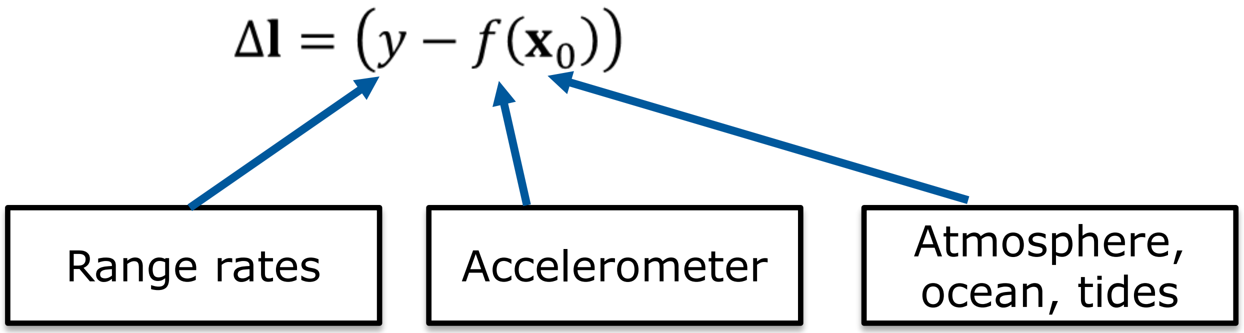

Gaussian filter 350 km


degree amplitudes (2010–02)

Summary of the first part

- Iterative algorithm to determine the weight matrix in least squares adjustment
  - Analysis of the post-fit residuals
  - Arcwise optimized relative weighting using variance component estimation (VCE) [Koch (1999): Parameter estimation and hypothesis testing in linear models, Springer, doi.org/10.1007/978-3-662-03976-2]
  - Ellmer (2018): Contributions to GRACE Gravity Field Recovery, doctoral thesis 2018, doi.org/10.3217/978-3-85125-646-8
  - Murböck et al. (2023): In-Orbit Performance of the GRACE Accelerometers and Microwave Ranging Instrument, Remote Sens. 2023, 15(3), 563; https://doi.org/10.3390/rs15030563

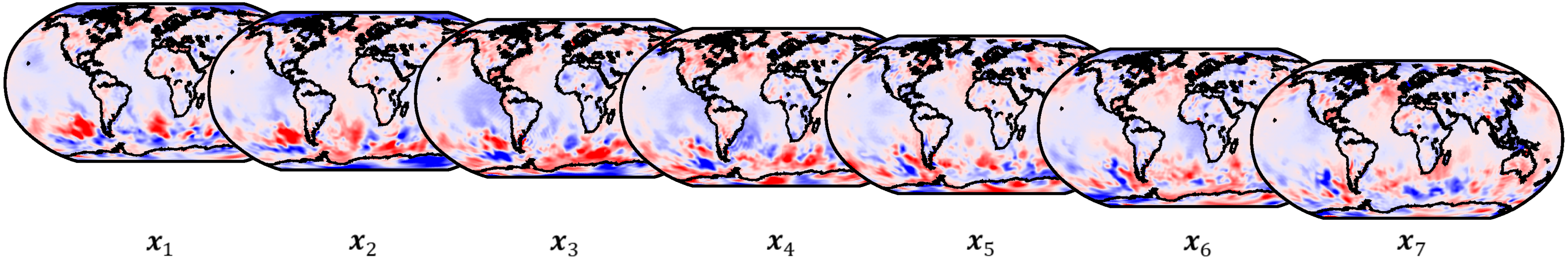
- Assumption: Noise time series is stationary (along the orbit)
  - This might be true for the instrument noise
  - But not for backgound model errors



$$\Delta\mathbf{l} = \left(y - f(\mathbf{x}_0)\right)$$

Background model noise

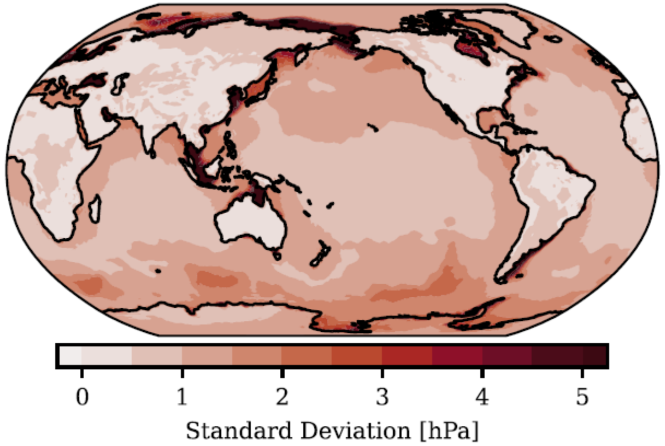
Shihora et al. (2024): Accounting for residual errors in atmosphere–ocean background models applied in satellite gravimetry, J Geod, 98:27, doi.org/10.1007/s00190-024-01832-7



Time series of vectors (SH coefficients), 26 years

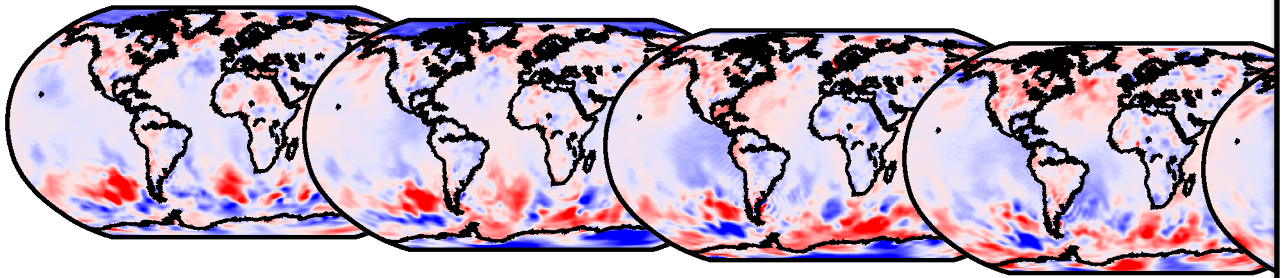
Assumption: stationary random process

$$\boldsymbol{\Sigma} = \frac{1}{N}\sum_{i=0}^{N} \boldsymbol{x}_i \mathbf{x}_i^T \qquad \boldsymbol{\Sigma}_{\Delta 1} = \frac{1}{N}\sum_{i=0}^{N} \boldsymbol{x}_{i-1} \mathbf{x}_i^T \qquad \boldsymbol{\Sigma}_{\Delta 2} = \frac{1}{N}\sum_{i=0}^{N} \boldsymbol{x}_{i-2} \mathbf{x}_i^T \qquad \cdots$$



Background model noise

Shihora et al. (2024): Accounting for residual errors in atmosphere–ocean background models applied in satellite gravimetry, Journal of Geodesy, 98:27, doi.org/10.1007/s00190-024-01832-7



Time series of vectors (SH coefficients), 26 yea

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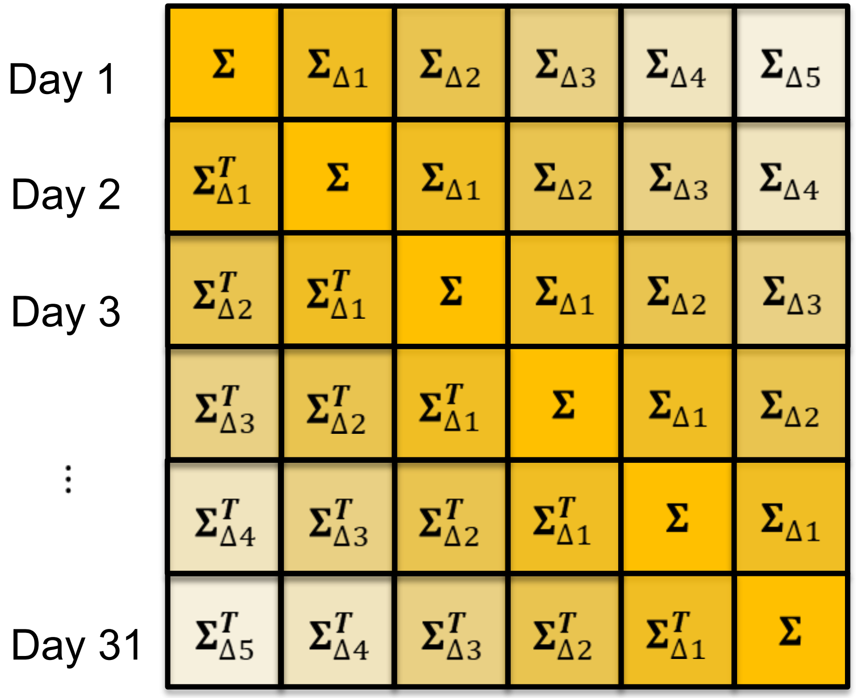
Full covariance matrix of one month
(Block Toeplitz matrix)



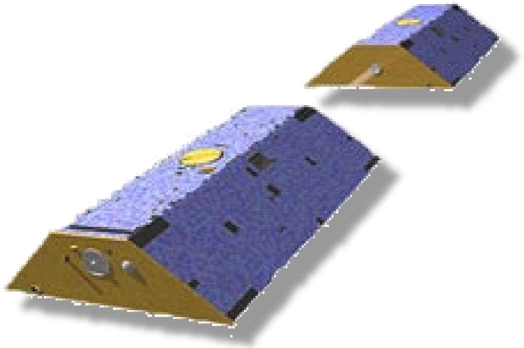
$$\begin{array}{c|cccccc}
\text{Day 1} & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} & \boldsymbol{\Sigma}_{\Delta 2} & \boldsymbol{\Sigma}_{\Delta 3} & \boldsymbol{\Sigma}_{\Delta 4} & \boldsymbol{\Sigma}_{\Delta 5} \\
\text{Day 2} & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} & \boldsymbol{\Sigma}_{\Delta 2} & \boldsymbol{\Sigma}_{\Delta 3} & \boldsymbol{\Sigma}_{\Delta 4} \\
\text{Day 3} & \boldsymbol{\Sigma}_{\Delta 2}^T & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} & \boldsymbol{\Sigma}_{\Delta 2} & \boldsymbol{\Sigma}_{\Delta 3} \\
 & \boldsymbol{\Sigma}_{\Delta 3}^T & \boldsymbol{\Sigma}_{\Delta 2}^T & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} & \boldsymbol{\Sigma}_{\Delta 2} \\
\vdots & \boldsymbol{\Sigma}_{\Delta 4}^T & \boldsymbol{\Sigma}_{\Delta 3}^T & \boldsymbol{\Sigma}_{\Delta 2}^T & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} \\
\text{Day 31} & \boldsymbol{\Sigma}_{\Delta 5}^T & \boldsymbol{\Sigma}_{\Delta 4}^T & \boldsymbol{\Sigma}_{\Delta 3}^T & \boldsymbol{\Sigma}_{\Delta 2}^T & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma}
\end{array}$$

Modeling the observation noise

Covariance matrix of the reduced observation vector

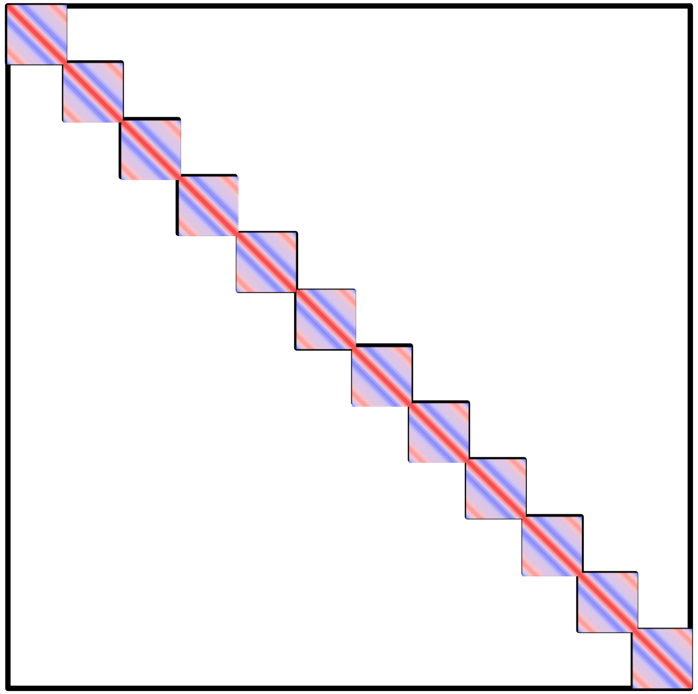
$$\boldsymbol{\Sigma}_{\Delta l} = \boldsymbol{\Sigma}_{instr} + \mathbf{B}\boldsymbol{\Sigma}_{model}\mathbf{B}^T$$

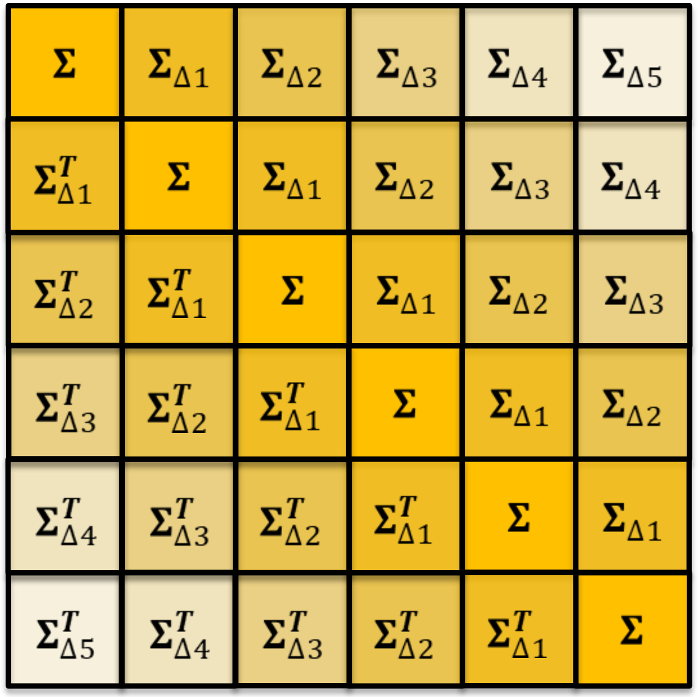
Variance propagation
(from spherical harmonics to range rates)



Stationary along orbit

Stationary at Earth surface





$$\begin{array}{l|cccccc}
\text{Day 1} & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} & \boldsymbol{\Sigma}_{\Delta 2} & \boldsymbol{\Sigma}_{\Delta 3} & \boldsymbol{\Sigma}_{\Delta 4} & \boldsymbol{\Sigma}_{\Delta 5} \\
\text{Day 2} & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} & \boldsymbol{\Sigma}_{\Delta 2} & \boldsymbol{\Sigma}_{\Delta 3} & \boldsymbol{\Sigma}_{\Delta 4} \\
\text{Day 3} & \boldsymbol{\Sigma}_{\Delta 2}^T & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} & \boldsymbol{\Sigma}_{\Delta 2} & \boldsymbol{\Sigma}_{\Delta 3} \\
\vdots & \boldsymbol{\Sigma}_{\Delta 3}^T & \boldsymbol{\Sigma}_{\Delta 2}^T & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} & \boldsymbol{\Sigma}_{\Delta 2} \\
 & \boldsymbol{\Sigma}_{\Delta 4}^T & \boldsymbol{\Sigma}_{\Delta 3}^T & \boldsymbol{\Sigma}_{\Delta 2}^T & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{\Delta 1} \\
\text{Day 31} & \boldsymbol{\Sigma}_{\Delta 5}^T & \boldsymbol{\Sigma}_{\Delta 4}^T & \boldsymbol{\Sigma}_{\Delta 3}^T & \boldsymbol{\Sigma}_{\Delta 2}^T & \boldsymbol{\Sigma}_{\Delta 1}^T & \boldsymbol{\Sigma}
\end{array}$$

Least squares adjustment

- Observation model

$\Delta\mathbf{l} = \mathbf{A}\,\Delta\mathbf{x} + \mathbf{e}$ with $\boldsymbol{\Sigma}_{\mathbf{e}} = \boldsymbol{\Sigma}_{instr} + \mathbf{B}\boldsymbol{\Sigma}_{model}\mathbf{B}^T$ $\mathbf{P}_{\mathbf{e}} = (\boldsymbol{\Sigma}_{instr} + \mathbf{B}\boldsymbol{\Sigma}_{model}\mathbf{B}^T)^{-1}$

Problem:
Matrix is fully occupied,
2 TB!

Least squares adjustment

- Observation model

$$\Delta\mathbf{l} = \mathbf{A}\,\Delta\mathbf{x} + \mathbf{e} \quad \text{with} \qquad \boldsymbol{\Sigma}_{\mathbf{e}} = \boldsymbol{\Sigma}_{instr} + \mathbf{B}\boldsymbol{\Sigma}_{model}\mathbf{B}^T \qquad \mathbf{P}_{\mathbf{e}} = (\boldsymbol{\Sigma}_{instr} + \mathbf{B}\boldsymbol{\Sigma}_{model}\mathbf{B}^T)^{-1}$$

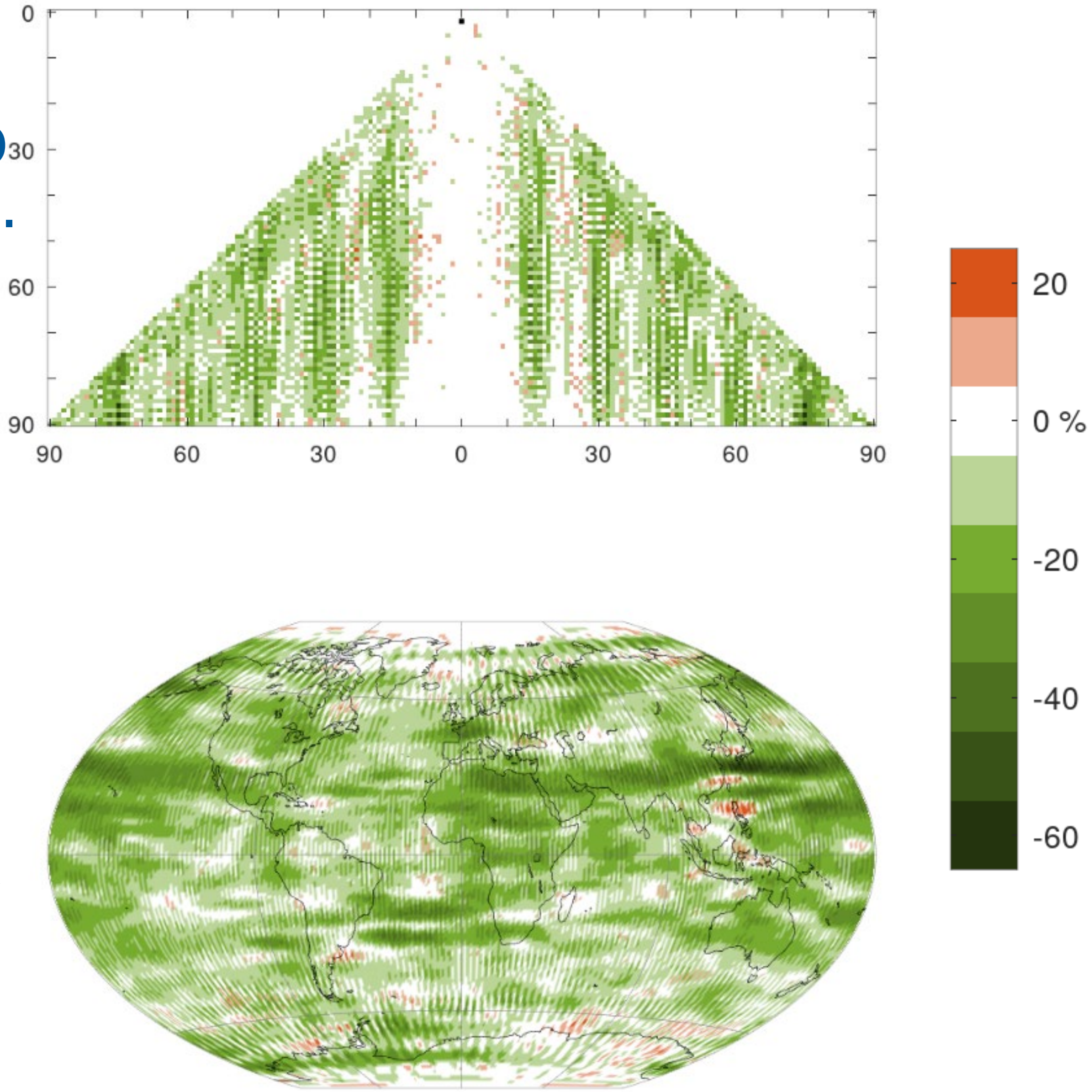
- Alternative model: co-estimation of submonthly (6 hourly) signals $\mathbf{y}$ and constraining these signals towards zero

$$\Delta\mathbf{l} = \mathbf{A}\,\Delta\mathbf{x} + \mathbf{B}\mathbf{y} + \mathbf{e} \qquad \boldsymbol{\Sigma}_{\mathbf{e}} = \boldsymbol{\Sigma}_{instr} \qquad \mathbf{P}_{\mathbf{e}} = \boldsymbol{\Sigma}_{instr}^{-1}$$

Both models are equivalent as shown in:
Kvas & Mayer-Gürr (2019): GRACE gravity field recovery with background model uncertainties. *J Geod* 93, 2543–2552. doi.org/10.1007/s00190-019-01314-1

Background model noise

Stochastic modeling applied for ocean tide and non-tidal AOD models improves GRACE/GRACE-FO gravity fields by up to 20 %.



Hauk et al. (2023). Satellite gravity field recovery using variance-covariance information from ocean tide models. Earth and Space Science, 10(10), e2023EA003098. https://doi.org/10.1029/2023EA003098

Wilms et al. (2025). Optimized gravity field retrieval for the MAGICmission concept using background model uncertainty information. *J Geod* 99.21. https://doi.org/10.1007/s00190-024-01931-5

Very short summary

- The weight matrix in least squares adjustment is important

$$||\Delta\mathbf{l} - \mathbf{A} \cdot \Delta\mathbf{x}||_{\mathbf{P}}^{2} \rightarrow \min$$

- Analyizing the post-fit residuals helps to derive a proper weight matrix

$$\hat{\mathbf{e}} = \Delta\mathbf{l} - \mathbf{A}\Delta\hat{\mathbf{x}}$$

- This is true for almost every least squares adjustment